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An Unstable Two-Phase Membrane Problem and Maximum Flux Exchange Flow

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Abstract Let U be a bounded open connected set in $\mathbb{R}^n$ $(n \geq 1)$. We refer to the unique weak solution of the Poisson problem $-\Delta u = \chi_A$ on U with Dirichlet boundary conditions as u_A for any measurable set A in U. The function $\psi := u_U$ is the torsion function of U. Let V be the measure $V := \psi\, \mathscr{L}^n$ on U where $\mathscr{L}^n$ stands for n-dimensional Lebesgue measure. We study the variational problem

$$I(U, p) := \sup \left\{ J(A) - V(U)\, p^2 \right\}$$

with $p \in (0, 1)$ where $J(A) := \int_A u_A \, dx$ and the supremum is taken over measurable sets $A \subset U$ subject to the constraint $V(A) = pV(U)$. We relate the above problem to an unstable two-phase membrane problem. We characterise optimsers in the case $n = 1$. The proof makes use of weighted isoperimetric and Pólya–Szegö inequalities.

Keywords Two-phase membrane problem · Isoperimetric inequality · Pólya–Szegö inequality · Spherical cap symmetrisation

Mathematics Subject Classification 26D10 · 35J20 · 35J60

1 Introduction and Motivation

Let U be a bounded open connected set in $\mathbb{R}^n$ $(n \geq 1)$. We refer to the unique weak solution of the Poisson problem

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$$
-\Delta u = \chi_A \text{ on } U,\ u \in W_0^{1,2}(U), \tag{1.1}
$$

as u_A for any measurable set A in U. The function $\psi := u_U$ is the torsion function of U. Let V be the measure $V := \psi\,\mathscr{L}^n$ on U where $\mathscr{L}^n$ stands for the n-dimensional Lebesgue measure. For $p \in (0,1)$ consider the variational problem

$$
I(U,p) := \sup\left\{J(A) - V(U)\,p^2\right\} \tag{1.2}
$$

where $J(A) := (u_A, \chi_A)$ and the supremum is taken over measurable sets $A \subset U$ subject to the constraint $V(A) = pV(U)$. Here, $(\cdot,\cdot)$ stands for the usual inner product in the real Hilbert space $L^2(U)$. Any maximiser E in (1.2) will be called an *optimal configuration* for the data (U,p). If E is an optimal configuration and $u = u_E$, then (u,E) will be called an *optimal pair*.

In Corollary 2.2 we show that for each $p \in (0,1)$ the problem (1.2) admits an optimal pair (u,E) for the data (U,p). In Proposition 3.3 we characterise the optimal configuration E as a super level set of u/ψ; that is, $E = \{u > c\psi\}$ for some $c \in (0,1)$ up to $\mathscr{L}^n$-a.e. equivalence. The derivation assumes that U is a $C^{1,1}$ domain. Under this last assumption, we show in Corollary 3.4 that u satisfies the following semi-linear elliptic partial differential equation with discontinuous nonlinearity. Put $v := u - c\psi$ with c as above. Then v is a strong solution of the problem

$$
-\Delta v = (1-c)\chi_{\Omega_+(v)} - c\chi_{\Omega_-(v)} \text{ on } U
$$

where $\Omega_\pm(v) := \{\pm v > 0\}$. The above equation is similar to Problem C (the two-phase membrane problem) in [20, 1.2.3] but with a sign change; see also the unstable membrane problem [20] 2.5. It is noted in [20, 1.1.7] that the composite membrane problem (see [6,7]) is akin to the unstable membrane problem. Our terminology is adopted from [6,7] and in places there is a similarity in method. The regular part of the free boundary $\Gamma(v) = \partial\Omega_\pm(v) \cap U$ is real-analytic (Theorem 3.7). In Sect. 4 we replace U with the unit ball B in $\mathbb{R}^n$ $(n \geq 2)$. For $p \in (0,1)$ we show that any optimal configuration E for the data (B,p) possesses spherical cap symmetry $\mathscr{L}^n$-a.e. (see Theorem 4.1).

In the remainder of the article, we study the problem (1.2) in the one-dimensional case $n = 1$ and take $B = (-1,1)$. In Theorem 9.5 we show that any optimal configuration E with data (B,p) is $\mathscr{L}^1$-a.e. equivalent to an open interval abutting a boundary point of B. A first step in obtaining this result is to transform the problem using an analog of the ground-state transformation (with the torsion function in place of the ground-state) (see Proposition 9.2). We then obtain an isoperimetric inequality on B with volume density ψ and perimeter density $\psi^{3/2}$ (Theorem 6.3) and a corresponding Hardy-Littlewood type inequality (Theorem 6.6) and a Pólya–Szegö inequality (Theorem 7.10). We also study the case of equality in the isoperimetric and Pólya–Szegö inequalities (Theorem 6.4 and Corollary 8.7 respectively). We have been guided by [2] in obtaining these results, though our setting and proofs are slightly different.

We have not obtained an analog of Theorem 9.5 in the case $n \geq 2$. At least part of our method transfers to higher dimensions. There is a counterpart of the isoperimetric

inequality Theorem 6.3 (though its derivation is more involved with the usual difficulties around regularity and stability) and the Hardy–Littlewood inequality is a ready consequence. A potential stumbling block is the validity of a corresponding Pólya–Szegö inequality. We note that the sufficient conditons given in [22] are stringent.
The problem (1.2) is related to maximum flux exchange flow (a model of magma flow in a volcanic vent [15]). We take $n = 2$ and consider a configuration of two immiscible fluids in a vertical duct with cross-section U in a state of steady flow. The densities of the fluids are labelled ρ, ρ' with $\rho > \rho'$ and each fluid has unit viscosity. The pressure p has constant gradient $\partial p/\partial z = -G$ on U. Suppose the fluid with density ρ occupies a region A in U. By the Navier–Stokes equations, the vertical component of velocity u satisfies

$$\begin{aligned} 0 &= \Delta u + G - \rho g \quad \text{on} \quad A; \\ 0 &= \Delta u + G - \rho' g \quad \text{on} \quad U \setminus A. \end{aligned}$$

Dirichlet boundary conditions are imposed on the boundary of U and it is assumed that u and its gradient are continuous on the interface between the two regions A and $U \setminus A$.
The parameter G lies in the interval $(\rho' g, \rho g)$ which allows the possibility of bidirectional flow. On rescaling (and relabelling the velocities) we obtain the system

$$\begin{aligned} 0 &= \Delta u - \lambda - 1 \quad \text{on} \quad A; \\ 0 &= \Delta u - \lambda + 1 \quad \text{on} \quad U \setminus A; \end{aligned} \tag{1.3}$$

where

$$\lambda := \frac{(\rho' + \rho)g - 2G}{(\rho - \rho')g} \in (-1, 1)$$

is a proxy for the pressure gradient. Two problems arise: one to maximise the flux $(\chi_{U \setminus A}, u)$ over regions A which satisfy the flux balance condition $(u, 1) = 0$ with constant λ; the other in which we optimize also over λ. In detail,

$$\gamma(U) := \sup \left\{ (\chi_{U \setminus A}, u) : (u, 1) = 0,\ A \subset U \text{ open},\ \lambda \in (-1, 1) \right\}, \tag{1.4}$$

$$\gamma(U, \lambda) := \sup \left\{ (\chi_{U \setminus A}, u) : (u, 1) = 0,\ A \subset U \text{ open} \right\}, \tag{1.5}$$

where in the latter λ is fixed in the interval $(-1, 1)$. In the case $n = 1$ and $U = B$ we show that any optimal configuration E for the problem (1.5) with data (B, λ) is $\mathscr{L}^1$-a.e. equivalent to an open interval abutting a boundary point of B in Theorem 9.8. Moreover, any optimal configuration E for the problem (1.4) is $\mathscr{L}^1$-a.e. equivalent to either $(-1, 0)$ or $(0, 1)$.

2 Existence of Optimal Configurations

Define

$$\mathscr{V}_t := \left\{ f \in L^2(U) : 0 \leq f \leq 1 \, \mathscr{L}^n\text{-a.e. on } U \text{ and } (f,\, \psi) \leq t \right\}$$

for $t \in (0, V(U))$ and consider the variational problem

$$\beta(U, t) := \sup \left\{ J(f) : f \in \mathscr{V}_t \right\} \tag{2.1}$$

where $J(f) := (u_f, f)$ and u_f is the unique solution of the Poisson problem (1.1) but with inhomogeneity f. The first main result runs as follows.

Theorem 2.1 *Fix $t \in (0, V(U))$. Then*

(i) *there exists $f \in \mathscr{V}_t$ such that $\beta(U, t) = J(f)$;*
(ii) *$(\psi, f) = t$;*
(iii) *f has the form $f = \chi_E$ for some measurable set E in U.*

Corollary 2.2 *For each $p \in (0, 1)$ the problem (1.2) admits an optimal pair (u, E) for the data (U, p).*

Proof Let $p \in (0, 1)$ and put $t := pV(U)$. Let E be as in Theorem 2.1 (iii). Then $V(E) = (\psi, \chi_E) = t = pV(U)$. Let $A \subset U$ be a measurable set with $V(A) = pV(U)$. Then $f = \chi_A \in \mathscr{V}_t$ so $J(E) \geq J(A)$. □

We prepare a few lemmas before proving Theorem 2.1.

Lemma 2.3 *Let X, Y be (real) Banach spaces and suppose that $X \subset Y$ with continuous embedding. Let (x_h) be a sequence in X which converges weakly in X to $x \in X$. Then (x_h) converges weakly to x in Y.*

Proof Note that for any $g \in Y'$, $g|_X \in X'$. □

We remark that the Dirichlet Laplacian $(D(\Delta),\, \Delta)$ is associated with the Dirichlet form $(\mathscr{F},\, \mathscr{E})$ in $L^2(U)$ with form domain $\mathscr{F} := W_0^{1,2}(U)$ and

$$\mathscr{E}(u,\, v) = \int_U \nabla u \cdot \nabla v \, dx \qquad (u, v \in \mathscr{F}).$$

Let G stand for the corresponding Green operator.

Lemma 2.4 *Let $t \in (0, V(U))$. Then*

(i) *the functional $J : \mathscr{V}_t \to \mathbb{R}$ is continuous in the topology of weak sequential convergence;*
(ii) *$J : \mathscr{V}_t \to \mathbb{R}$ is convex.*

Proof (i) Suppose that a sequence (f_h) in $\mathscr{V}_t$ converges weakly to an element $f \in \mathscr{V}_t$ in $L^2(U)$. Put $u_h := Gf_h \in L^2(U)$. For each h and $\varphi \in L^2(U)$,

$$(u_h,\ \varphi) = (Gf_h,\ \varphi) = (f_h,\ G\varphi),$$

by symmetry of G so that $(u_h,\ \varphi) \to (f,\ G\varphi) = (u,\ \varphi)$ as $h \to \infty$ where $u := Gf$. We also have that

$$\mathscr{E}(u_h,\ \varphi) = (f_h,\ \varphi) \to (f,\ \varphi) = \mathscr{E}(u,\ \varphi) \text{ as } h \to \infty$$

for any $\varphi \in \mathscr{F}$. This means that (u_h) converges weakly to u in $\mathscr{F}$.
Note that $W_0^{1,2}(U) \subset W_0^{1,1}(U)$ and $\|u\|_{W_0^{1,1}(U)} \leq \sqrt{2\,|U|}\,\|u\|_{W_0^{1,2}(U)}$ for each $u \in W_0^{1,2}(U)$. By Lemma 2.3, (u_h) converges weakly to u in $W_0^{1,1}(U)$.
If $n \geq 2$ we may use the Rellich–Kondrachov compactness theorem [10, 5.7] for example and [16, Theorem 21.2.9] to conclude that (u_h) converges strongly to u in $L^1(U)$. Now

$$J(f) - J(f_h) = (u,\ f) - (u_h,\ f_h) = (u,\ f - f_h) + (f_h, u - u_h)$$

and the right-hand side converges to zero as $h \to \infty$ as (f_h) is bounded in $L^\infty(U)$. This shows that $J(f_h) \to J(f)$ as $h \to \infty$.
In the case $n = 1$ we use the fact that $W_0^{1,2}(U)$ is compactly embedded in $C^0(\overline{U})$ (see [13, Theorem 7.22]) and hence in $L^1(U)$.
(ii) Let $f \in \mathscr{V}_t$. By Dirichlet's principle [10, 2.2.5] for example,

$$E(f) := \inf_{v \in \mathscr{F}} \Big\{(1/2)\,\mathscr{E}(v,\ v) - (v,\ f)\Big\} = -(1/2)\,J(f).$$

The functional E is concave so that J is convex. □

Lemma 2.5 *Let $t \in (0,\ V(U))$. A function f in the convex set $\mathscr{V}_t \subset L^2(U)$ is extremal only if $f = \chi_A$ $\mathscr{L}^n$-a.e. on U for some $A \subset U$ measurable with $(\psi,\ \chi_A) \leq t$.*

Proof The proof runs as in [11, Lemma 2]. A measurable function f on U is $\mathscr{L}^n$-a.e. equivalent to χ_A for some $A \subset U$ measurable if and only if $f(1-f) = 0$ $\mathscr{L}^n$-a.e. on U. Suppose that $f \in \mathscr{V}_t$ is an extremal element and assume that $|\{f(1-f) \neq 0\}| > 0$ for a contradiction. Then there exists $\varepsilon > 0$ and a measurable set E in U with positive $\mathscr{L}^n$-measure such that $\varepsilon \leq f \leq 1 - \varepsilon$ on E. Decompose E into two disjoint sets $E_1,\ E_2$ each with positive $\mathscr{L}^n$-measure. Choose $\alpha = (\alpha_1,\ \alpha_2) \in \mathbb{R}^2 \setminus \{0\}$ such that $\alpha_1(\psi,\ \chi_{E_1}) + \alpha_2(\psi,\ \chi_{E_2}) = 0$ and define

$$f_\tau := f + \tau \sum_{j=1}^{2} \alpha_j \chi_{E_j}$$

for $\tau \in \mathbb{R}$. Then $f_\tau \in \mathscr{V}_t$ for $|\tau| \leq \varepsilon/|\alpha_1| \vee |\alpha_2|$. We then derive the contradiction that f is not extremal as $f = (1/2)\,\{f_\tau + f_{-\tau}\}$ for such τ. □

Proof of Theorem 2.1 Let (f_h) be a maximising sequence for $\beta(U,t)$. Now $\mathscr{V}_t$ is weakly sequentially compact in $L^2(U)$. This follows by appeal to [16, Theorem 10.2.9] due to the fact that $\mathscr{V}_t$ is bounded, closed and convex in the reflexive Banach space $L^2(U)$. So we may assume that (f_h) converges weakly in $L^2(U)$ to some $f \in \mathscr{V}_t$ as $h \to \infty$ after choosing a subsequence if necessary. By Lemma 2.4 (i),

$$\beta(U,t) = \lim_{h\to\infty} J(f_h) = J(f),$$

giving item (i) of the Theorem. It is straightforward to see that $(\psi,\ f) = t$ and hence (ii).

We now argue as in [8, Corollary 6.2]. By [5, Chapitre II §7 Proposition 1.1 (EVT II.58)], J attains its supremum on $\mathscr{V}_t$ at an extremal point f. We then invoke Lemma 2.5 to conclude that f has the form $f = \chi_A$ $\mathscr{L}^n$-a.e. on U for some measurable set A in U and hence (iii). □

3 Some Partial Regularity Results

Proposition 3.1 *Suppose that U is a $C^{1,1}$ domain. Let $E \subset U$ be a measurable set and $u = u_E$. Then $V(\{u = t\psi\}) = 0$ for each $t \in (0,1)$.*

Proof By [13, Theorem 9.15], $u \in W^{2,p}(U)$ for any $1 < p < \infty$. Put $v := u - t\psi$, $N_t := \{v = 0\}$ and assume that $|N_t| > 0$. By [13, Lemma 7.7], we derive that $D^\alpha v = 0$ $\mathscr{L}^n$-a.e. and hence V-a.e. on N_t for any multi-index α with $|\alpha| = 1$. Observe that $D^\alpha v$ belongs to $W^{1,p}(U)$ for $|\alpha| \leq 1$. Applying the last-mentioned lemma once more, we see $D^\alpha v = 0$ V-a.e. on N_t for any multi-index α with $|\alpha| \leq 2$. So $-\Delta v = 0$ V-a.e. on N_t. But $-\Delta v = \chi_E - t\,\chi_U$ V-a.e. on U. This leads to a contradiction. □

We require a version of the bathtub principle (see [17, Theorem 1.14]). Let $(X, \mathscr{A}, \mu)$ be a finite measure space and ρ a positive $\mathscr{A}$-measurable integrable function on X. Given $0 < v < \mu(X)$, consider the variational problem

$$\sup \int_X \chi_E\, \rho\, d\mu \tag{3.1}$$

where the supremum is taken over measurable sets $E \subset X$ with $\mu(E) = v$. We say that measurable sets A, B in X are equivalent μ-a.e. and write $A = B$ if and only if $\mu(A \Delta B) = 0$.

Theorem 3.2 *Assume that*

$$\mu(\{\rho = t\}) = 0 \textit{ for all } t > 0. \tag{3.2}$$

Then for each $v \in (0, \mu(X))$ the problem (3.1) has a unique optimiser up to equivalence μ-a.e. given by $E = \{\rho > s\}$ where

$$s := \inf\{\tau > 0 : \mu(\{\rho > \tau\}) \leq v\}.$$

Proof The distribution function $\mu_\rho : (0, \infty) \to (0, V(U)); \tau \mapsto \mu(\{\rho > \tau\})$ is non-increasing and right-continuous on $(0, \infty)$; in fact, continuous thanks to (3.2). By right-continuity of μ_ρ, $\mu_\rho(s) \le v$; by left-continuity, the reverse inequality holds, so $\mu(E) = \mu_\rho(s) = v$. For a measurable set A in X with $\mu(A) = v$,

$$\begin{aligned}\int_X \chi_A \, \rho \, d\mu &= \int_0^\infty \mu(A \cap \{\rho > \tau\}) \, d\tau \\ &= \int_0^s \mu(A \cap \{\rho > \tau\}) \, d\tau + \int_s^\infty \mu(A \cap \{\rho > \tau\}) \, d\tau \\ &\le sv + \int_s^\infty \mu(\{\rho > \tau\}) \, d\tau = \int_X \chi_E \, \rho \, d\mu\end{aligned}$$

according to the layer cake representation [17, Theorem 1.13]. It follows that E is an optimiser for (3.1).
Suppose A is a measurable set in X with $\mu(A) = v$ which is not μ-a.e. equivalent to E. Then

$$\mu(E) = \mu(E \setminus A) + \mu(E \cap A) = v = \mu(A \setminus E) + \mu(E \cap A) = \mu(A)$$

so $\mu(E \setminus A) > 0$ as otherwise $\mu(A \setminus E) = 0$ and A is μ-a.e. equivalent to E. By countable additivity,

$$0 < \mu(E \setminus A) = \lim_{\tau \downarrow s} \mu(\{\rho > \tau\} \setminus A). \tag{3.3}$$

Thus,

$$\begin{aligned}&\int_X \chi_A \, \rho \, d\mu \\ &= \int_0^s \mu(A \cap \{\rho > \tau\}) \, d\tau + \int_s^\infty \mu(\{\rho > \tau\}) \, d\tau - \int_s^\infty \mu(\{\rho > \tau\} \setminus A) \, d\tau \\ &\le s\, v + \int_s^\infty \mu(\{\rho > \tau\}) \, d\tau - \int_s^\infty \mu(\{\rho > \tau\} \setminus A) \, d\tau \\ &= \int_X \chi_E \, \rho \, d\mu - \int_s^\infty \mu(\{\rho > \tau\} \setminus A) \, d\tau < \int_X \chi_E \, \rho \, d\mu\end{aligned}$$

where the strict inequality follows from (3.3). □

Let U be a $C^{1,1}$ domain and $p \in (0, 1)$. Let (u, E) be an optimal pair for (1.2) with data (U, p). By [13, Corollary 9.18] we may assume that $u \in C^0(\overline{U})$.

Proposition 3.3 *Suppose that U is a $C^{1,1}$ domain. Let $p \in (0, 1)$ and suppose that (u, E) is an optimal pair for (1.2) with data (U, p). Then $V(E \Delta \{u > c\psi\}) = V(E \Delta \{u \ge c\psi\}) = 0$ where $c \in (0, 1)$ is uniquely determined by the condition*

$$V(\{u > c\psi\}) = pV(U). \tag{3.4}$$

Proof Put $F := \{u > c\psi\}$ with c as in (3.4). Assume for a contradiction that $V(E\Delta F) > 0$. We consider a version of Problem (3.1) on U with ρ replaced by $w := u_E/\psi$ and μ replaced by V. By Proposition 3.1, $V(\{w = t\}) = 0$ for each $t > 0$; thus condition (3.2) holds. By uniqueness of the optimiser in Theorem 3.2 and the Cauchy–Schwarz inequality,

$$J(E) = (u_E, \chi_E) = \int_U w\chi_E\, dV < \int_U w\chi_F\, dV = \mathscr{E}(u_E, u_F) \leq J(E)^{1/2} J(F)^{1/2}$$

so that $J(E) < J(F)$, contradicting the assumption that E is an optimal configuration. The identity $V(E\Delta\{u \geq c\psi\}) = 0$ follows from Proposition 3.1. □

Corollary 3.4 *Suppose that U is a $C^{1,1}$ domain. Let $p \in (0, 1)$ and suppose that (u, E) is an optimal pair for (1.2) with data (U, p). Put $v := u - c\psi$ where c is given by (3.4). Then v is a strong solution of the problem*

$$-\Delta v = (1-c)\chi_{\Omega_+(v)} - c\chi_{\Omega_-(v)} \text{ on } U \tag{3.5}$$

where $\Omega_\pm(v) := \{\pm v > 0\}$ and $E = \Omega_+(v)$ $\mathscr{L}^n$-a.e.

Proof By [13, Theorem 9.15], $u \in W^{2,p}(U)$ for any $1 < p < \infty$ and u is a strong solution of $-\Delta u = \chi_E$. By Proposition 3.3, u is a strong solution of $-\Delta u = \chi_{\{u>c\psi\}}$. The result follows from the fact that $-\Delta(c\psi) = c\chi_U$ and Proposition 3.1. □

Lemma 3.5 *Let $p \in (0, 1)$ and (u, E) be an optimal pair for the data (U, p). Then $(\psi - u, U \setminus E)$ is an optimal pair for the data $(U, 1 - p)$.*

Proof Let $A \subset U$ be a measurable set with $V(A) = pV(U)$. Then

$$J(U \setminus A) = J(A) - V(A) + V(U \setminus A) = J(A) + (1 - 2p)V(U)$$

so that $J(U \setminus A) - V(U)(1 - p)^2 = J(A) - V(U)p^2$ and the result follows. □

Put $\Gamma_\pm(v) = \partial\Omega_\pm(v) \cap U$.

Lemma 3.6 *Suppose that U is a $C^{1,1}$ domain. Suppose that (u, E) is an optimal pair for the data (U, p) and let v be as in Corollary 3.4. Then $\Gamma_+(v) = \Gamma_-(v)$.*

Proof Suppose that $x \in \Gamma_+(v)\setminus\Gamma_-(v)$. Then there exists $r > 0$ such that $B(x, r) \subset U$, $u \geq c\,\psi$ on $B(x, r)$ and $u(x) = c\psi(x)$. By Proposition 3.3, $V(B(x, r)\setminus E) \leq V(\{u \geq c\psi\} \setminus E) = 0$ and $B(x, r) \setminus E$ is a Lebesgue null set. Let Φ stand for the fundamental solution of Laplace's equation in $\mathbb{R}^n$. By the mean-value formula (see [10, 2.5 Problem 3] for example), for any $0 < \tau < r$,

$$
\begin{aligned}
u(x) &= |B(x,\tau)|^{-1}\int_{\partial B(x,\tau)} u\, d\mathcal{H}^{n-1} + \int_{B(x,\tau)} \Big\{\Phi(y-x)-\Phi(\tau\omega)\Big\}\chi_E\, dy \\
&= |B(x,\tau)|^{-1}\int_{\partial B(x,\tau)} u\, d\mathcal{H}^{n-1} + \int_{B(x,\tau)} \Big\{\Phi(y-x)-\Phi(\tau\omega)\Big\}\, dy \\
&\geq c\,|B(x,\tau)|^{-1}\int_{\partial B(x,\tau)} \psi\, d\mathcal{H}^{n-1} + \int_{B(x,\tau)} \Big\{\Phi(y-x)-\Phi(\tau\omega)\Big\}\, dy \\
&> c\left\{|B(x,\tau)|^{-1}\int_{\partial B(x,\tau)} \psi\, d\mathcal{H}^{n-1} + \int_{B(x,\tau)} \Big\{\Phi(y-x)-\Phi(\tau\omega)\Big\}\, dy\right\} \\
&= c\psi(x)
\end{aligned}
$$

as $c \in (0,1)$, a contradiction. Here, ω is an arbitrary element in the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.

Now suppose that $x \in \Gamma_-(v)\setminus\Gamma_+(v)$. As before, there exists $r > 0$ such that $u \leq c\,\psi$ on $B(x,r)$ and $u(x) = c\psi(x)$; alternatively, $\psi - u \geq (1-c)\,\psi$ on $B(x,r)$ and $(\psi-u)(x) = (1-c)\psi(x)$. By Lemma 3.5, $(\psi - u, U\setminus E)$ is an optimal pair for the data $(U, 1-p)$. We then get a contradiction as above. □

Put $\Gamma(v) := \Gamma_+(v) = \Gamma_-(v)$ and $\Gamma^*(v) := \Gamma(v)\cap\{|\nabla v| \neq 0\}$. The next theorem follows as in [20, Theorem 4.24].

Theorem 3.7 *Suppose that U is a $C^{1,1}$ domain. Suppose that (u, E) is an optimal pair for the data (U, p) and that $x_0 \in \Gamma^*(v)$. Then there exists $r > 0$ such that $\Gamma(v)\cap B(x,r)$ is a real-analytic hypersurface in $B(x,r)$.*

4 Spherical Cap Symmetry

In this section, we replace U by the open unit ball B in $\mathbb{R}^n$ ($n \geq 2$) centred at the origin. We prove the following symmetry result. The notion of spherical cap symmetry is defined below.

Theorem 4.1 *Let $p \in (0,1)$. Suppose that (u, E) is an optimal pair for the data (B, p). Then E possesses spherical cap symmetry $\mathscr{L}^n$-a.e.*

We first discuss the operation of polarisation for integrable functions on B (see [4] and references therein). For $\nu \in \mathbb{S}^{n-1}$ the closed half-space $H = H_\nu$ is defined by

$$
H_\nu := \Big\{x \in \mathbb{R}^n : x\cdot\nu \geq 0\Big\}
$$

with an associated reflection $\tau_H : \mathbb{R}^n \to \mathbb{R}^n$; $x \mapsto x - 2\,(x\cdot\nu)\,\nu$. We refer to the collection of closed half-spaces H by $\mathscr{H}$. The polarisation f_H of $f \in L^1_+(B)$ with respect to $H \in \mathscr{H}$ is defined as follows. Choose an $\mathscr{L}^n$-version $\widetilde{f}$ of f. Set

$$
\widetilde{f}_H(x) := \begin{cases} \widetilde{f}(x)\vee\widetilde{f}(\tau_H x) & \text{for } x \in B\cap H, \\ \widetilde{f}(x)\wedge\widetilde{f}(\tau_H x) & \text{for } x \in B\setminus H; \end{cases}
$$

$\widetilde{f}_H$ is $\mathscr{L}^n$-measurable and its $\mathscr{L}^n$-equivalence class $f_H := [\widetilde{f}_H]$ is the polarisation of f. The definition is well-defined due to the fact that if $\widetilde{f} = \widetilde{g}$ $\mathscr{L}^n$-a.e. on B then $\widetilde{f} = \widetilde{g}$ $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}_\tau$ for $\mathscr{L}^1$-a.e. $0 < \tau < 1$, and vice-versa.
The Green kernel $G(x, y)$ for B is given by

$$G(x, y) = \Phi(y - x) - \Phi(|x|(y - x^*)) \text{ for } (x, y) \in B \times B \setminus \mathtt{d},$$

where Φ is the fundamental solution of Laplace's equation in $\mathbb{R}^n$ as before, $\mathtt{d}$ stands for the diagonal in $B \times B$ and the decoration $*$ refers to inversion in the unit sphere. We note the inequality

$$G(x, y) > G(\tau_H x, y) \text{ for any } x, y \in B \cap \operatorname{int} H, \tag{4.1}$$

which follows from the strong maximum principle.

Theorem 4.2 *Let $f \in L^1_+(B)$ and $H \in \mathscr{H}$. Then $J(f) \le J(f_H)$ with equality if and only if either $f = f_H$ or $f \circ \tau_H = f_H$ $\mathscr{L}^n$-a.e. on B.*

Proof Let $\widetilde{f}$ be an $\mathscr{L}^n$-version of f. Define

$$A^+ := \left\{x \in B \cap H : \widetilde{f}(x) < \widetilde{f}(\tau_H x)\right\}$$

and similarly S^+ but with the strict inequality replaced by the sign $>$. Put $A^- := \tau_H A^+$ and $A := A^+ \cup A^-$. In this notation,

$$\widetilde{f}_H = \chi_A \widetilde{f} \circ \tau_H + \chi_{B\setminus A} \widetilde{f}.$$

As a consequence,

$$\begin{aligned} J(f_H) &= J(\chi_A\, \widetilde{f} \circ \tau_H) + 2\, (\chi_A \widetilde{f} \circ \tau_H, G\chi_{B\setminus A} \widetilde{f}) + J(\chi_{B\setminus A} \widetilde{f}) \\ &= J(\chi_A \widetilde{f}) + 2(\chi_A \widetilde{f} \circ \tau_H, G\chi_{B\setminus A} \widetilde{f}) + J(\chi_{B\setminus A} \widetilde{f}) \end{aligned}$$

and a similar identity holds for $J(f)$ but without composition with reflection. We may then write

$$\begin{aligned} &J(f_H) - J(f) \\ &= 2(\chi_A[\widetilde{f} \circ \tau_H - \widetilde{f}], G\chi_{B\setminus A} \widetilde{f}) \\ &= 2\int_{A^+} \int_{B\cap H\setminus A^+} (\widetilde{f}(\tau_H x) - \widetilde{f}(x)) G(x, y) \widetilde{f}(y)\, dy dx \\ &\quad + 2\int_{A^+} \int_{(B\setminus H)\setminus A^-} (\widetilde{f}(\tau_H x) - \widetilde{f}(x)) G(x, y) \widetilde{f}(y)\, dy dx \\ &\quad + 2\int_{A^-} \int_{B\cap H\setminus A^+} (\widetilde{f}(\tau_H x) - \widetilde{f}(x)) G(x, y) \widetilde{f}(y)\, dy dx \\ &\quad + 2\int_{A^-} \int_{(B\setminus H)\setminus A^-} (\widetilde{f}(\tau_H x) - \widetilde{f}(x)) G(x, y) \widetilde{f}(y)\, dy dx \end{aligned}$$

$$
\begin{aligned}
&= 2\int_{A^+}\int_{B\cap H\setminus A^+} (\widetilde{f}(\tau_H x) - \widetilde{f}(x))G(x,y)\widetilde{f}(y)\,dydx \\
&\quad + 2\int_{A^+}\int_{B\cap H\setminus A^+} (\widetilde{f}(\tau_H x) - \widetilde{f}(x))G(\tau_H x,y)\widetilde{f}(\tau_H y)\,dydx \\
&\quad - 2\int_{A^+}\int_{B\cap H\setminus A^+} (\widetilde{f}(\tau_H x) - \widetilde{f}(x))G(x,\tau_H y)\widetilde{f}(y)\,dydx \\
&\quad - 2\int_{A^+}\int_{B\cap H\setminus A^+} (\widetilde{f}(\tau_H x) - \widetilde{f}(x))G(x,y)\widetilde{f}(\tau_H y)\,dydx \\
&= 2\int_{A^+}\int_{B\cap H\setminus A^+} (\widetilde{f}(\tau_H x) - \widetilde{f}(x))(G(x,y) - G(\tau_H x,y)\widetilde{f}(y)\,dydx \\
&\quad - 2\int_{A^+}\int_{B\cap H\setminus A^+} (\widetilde{f}(\tau_H x) - \widetilde{f}(x))(G(x,y) - G(\tau_H x,y)\widetilde{f}(\tau_H y)\,dydx \\
&= 2\int_{A^+}\int_{S^+} (\widetilde{f}(\tau_H x) - \widetilde{f}(x))(G(x,y) - G(\tau_H x,y))(\widetilde{f}(y) - \widetilde{f}(\tau_H y))\,dydx.
\end{aligned}
$$

It is clear from this representation with the help of (4.1) that $J(f) \leq J(f_H)$. In the case of equality, it holds that either $|A^+| = 0$ or $|S^+| = 0$. In the former case, $f = f_H$ while in the latter, $f \circ \tau_H = f_H$ $\mathscr{L}^n$-a.e. on B. $\square$

Let $\omega \in \mathbb{S}^{n-1}$. Given $0 < \tau < 1$ and $0 < \alpha \leq \pi$ the spherical cap $C_\omega(\tau, \alpha)$ is the set

$$
C_\omega(\tau,\alpha) := \left\{x = \tau\cos\theta\,\omega + \tau\sin\theta\,\eta : 0 \leq \theta < \alpha, \eta \in \mathbb{S}^{n-1}\cap\omega^\perp\right\} \subset \mathbb{S}^{n-1}_\tau
$$

and has volume

$$
s(\tau,\alpha) := \mathcal{H}^{n-1}(S_\omega(\tau,\alpha)) = \omega_{n-2}\tau^{n-1}\int_0^\alpha (\sin\theta)^{n-2}\,d\theta.
$$

For a Borel set E in B put

$$
L(\tau) := \mathcal{H}^{n-1}(E \cap \mathbb{S}^{n-1}_\tau) \text{ for } 0 \leq \tau < 1 \text{ and } p(E) := \{0 \leq \tau < 1 : L(\tau) > 0\}.
$$

The function L is Borel measurable The spherical cap symmetrisation of E is the set

$$
C_\omega E := \bigcup_{\tau\in p(E)} C_\omega(\tau,\alpha) \tag{4.2}
$$

where $\alpha \in (0, \pi]$ is determined by $s(\tau, \alpha) = L(\tau)$. Observe that $C_\omega E$ is a Borel set in B (use Fubini's Theorem [1, 1.74] for example) and $|C_\omega E| = |E|$. We say that the Borel set $E \subset B$ possesses spherical cap symmetry $\mathscr{L}^n$-a.e. if $C_\omega E = E$ up to $\mathscr{L}^n$-a.e. equivalence for some $\omega \in \mathbb{S}^{n-1}$.
Let $f \in L^1_+(B)$ and choose an $\mathscr{L}^n$-version $\widetilde{f}$ of f. Put $m_{\widetilde{f}}(\tau,t) := \mathcal{H}^{n-1}(\{\widetilde{f} > t\}\cap\mathbb{S}^{n-1}_\tau)$ for $t \in \mathbb{R}$ and $0 \leq \tau < 1$. The function $m_{\widetilde{f}}(\tau,\cdot)$ is non-increasing and right continuous. Define its right continuous inverse by

$$\widetilde{f}^\sharp(\tau, s) := \inf\{t \in \mathbb{R} : m_{\widetilde{f}}(\tau, t) \leq s\} \text{ for } 0 < s \leq \mathcal{H}^{n-1}(\mathbb{S}^{n-1}_\tau).$$

For $x \in B$ put $\tau = |x|$ and choose $\alpha \in (0, \pi]$ such that $x \cdot \omega = \tau \cos\alpha$ then define

$$C_\omega \widetilde{f}(x) := \widetilde{f}^\sharp(\tau, s(\tau, \alpha)).$$

Note that $m_{\widetilde{f}}(\tau, t) > s$ if and only if $\widetilde{f}^\sharp(\tau, s) > t$. It follows that

$$\left\{C_\omega \widetilde{f} > t\right\} = C_\omega \left\{\widetilde{f} > t\right\} \text{ for each } t \in \mathbb{R}. \tag{4.3}$$

In particular, $C_\omega \widetilde{f}$ is Borel measurable and its $\mathscr{L}^n$-equivalence class $C_\omega f := [C_\omega \widetilde{f}]$ is the spherical cap symmetrisation of f.

Before proving Theorem 4.1, we prepare a number of lemmas. We first discuss a useful two-point inequality. We introduce the notation

$$\begin{aligned} Q &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0 \text{ and } x_2 \geq 0\right\}, \\ R &:= \{(x_1, x_2) \in Q : 0 \leq x_2 < x_1\}, \\ S &:= \{(x_1, x_2) \in Q : 0 \leq x_1 < x_2\}. \end{aligned}$$

Equip Q with the ℓ^1-norm $\|x\|_1 := |x_1| + |x_2|$ and define a mapping $\varphi : Q \to Q$ via $(x_1,\ x_2) \mapsto (x_1 \vee x_2,\ x_1 \wedge x_2)$; φ folds S onto R. A geometric argument establishes the following lemma.

Lemma 4.3 *For any $x,\ y \in Q$, $\|\varphi x - \varphi y\|_1 \leq \|x - y\|_1$ with strict inequality if and only if $x \in R$ and $y \in \overline{S}$ or $x \in \overline{R}$ and $y \in S$ or the same with the rôles of x and y interchanged.*

For $\omega \in \mathbb{S}^{n-1}$ introduce the collection of closed half-spaces $\mathscr{H}_\omega := \Big\{H_\nu :\ \nu \in \mathbb{S}^{n-1}$ and $\nu \cdot \omega \geq 0\Big\}$.

Lemma 4.4 *Fix $\omega \in \mathbb{S}^{n-1}$. For any $H \in \mathscr{H}_\omega$ we have*

(i) *for any $f, g \in L^1_+(B)$, $\|f_H - g_H\|_{L^1(B)} \leq \|f - g\|_{L^1(B)}$;*
(ii) *for any $f \in L^1_+(B)$, $(C_\omega f)_H = C_\omega f$ $\mathscr{L}^n$-a.e. on B;*
(iii) *for any $f \in L^1_+(B)$,*

$$\|f_H - C_\omega f\|_{L^1(B)} \leq \|f - C_\omega f\|_{L^1(B)} \tag{4.4}$$

with strict inequality if $|\{f \circ \tau_H > f\} \cap H| > 0$.

Proof (i) By Lemma 4.3,

$$
\begin{aligned}
&\|f_H - g_H\|_{L^1(B)}\\
&= \int_{B\cap H} |f_H - g_H|\,dx + \int_{B\setminus H} |f_H - g_H|\,dx\\
&= \int_{B\cap H} |f \vee (f\circ\tau_H) - g \vee (g\circ\tau_H|\,dx + \int_{B\setminus H} |f \wedge (f\circ\tau_H) - g \wedge (g\circ\tau_H|\,dx\\
&= \int_{B\cap H} \Big\{ |f \vee (f\circ\tau_H) - g \vee (g\circ\tau_H| + |(f\circ\tau_H)\wedge f - (g\circ\tau_H)\wedge g| \Big\}\,dx\\
&= \int_{B\cap H} \|\varphi(f, f\circ\tau_H) - \varphi(g, g\circ\tau_H)\|_1\,dx\\
&\le \int_{B\cap H} \|(f, f\circ\tau_H) - (g, g\circ\tau_H)\|_1\,dx\\
&= \int_{B\cap H} \Big\{ |f-g| + |f\circ\tau_H - g\circ\tau_H| \Big\}\,dx = \|f-g\|_{L^1(B)}.
\end{aligned}
$$

(ii) Let $f \in L^1_+(B)$ and $\widetilde{f}$ a $\mathscr{L}^n$-representative of f. For $x \in B\cap H$, $x\cdot\omega \ge (\tau_H x)\cdot\omega$ so $(C_\omega \widetilde{f})_H = C_\omega \widetilde{f}$ on B. Therefore,

$$
C_\omega f = [C_\omega \widetilde{f}] = [(C_\omega \widetilde{f})_H] = [C_\omega \widetilde{f}]_H = (C_\omega f)_H.
$$

(iii) The inequality follows by (i) and (ii). On $B\cap H$ the pair $(C_\omega \widetilde{f}, C_\omega \widetilde{f}\circ\tau_H)$ belongs to $\overline{R}$. By Lemma 4.3 if $(\widetilde{f}, \widetilde{f}\circ\tau_H) \in S$ on a set of positive measure in $B\cap H$ then strict inequality holds in (4.4). This observation leads to the criterion in the lemma. □

Lemma 4.5 *Let $f \in L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1})$. Fix $\nu \in \mathbb{S}^{n-1}$ and let (ν_h) be a sequence in $\mathbb{S}^{n-1}$ that converges to ν in $\mathbb{S}^{n-1}$. Then*

$$
\|f\circ\tau_h - f\circ\tau\|_{L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1})} \to 0 \;\; \text{as } h \to \infty.
$$

Proof Note that $|\tau_h x - \tau x| \le 4|\nu_h - \nu|$ for each $x \in \mathbb{S}^{n-1}$ and h. Now use the density of $C(\mathbb{S}^{n-1})$ in $L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1})$ and the fact that each τ, τ_h is an isometry on $L^1(\mathbb{S}^{n-1}, \mathcal{H}^{n-1})$. □

The next lemma is a spherical cap symmetrisation counterpart to [4, Lemma 6.3] and extends [23, Lemma 3.9].

Lemma 4.6 *Let $f \in L^1_+(B)$ and $\omega \in \mathbb{S}^{n-1}$ and assume that $f \ne C_\omega f$. Then there exists $H \in \mathscr{H}_\omega$ such that*

$$
\|f_H - C_\omega f\|_{L^1(B)} < \|f - C_\omega f\|_{L^1(B)}.
$$

Proof For non-negative Borel measurable functions f, g on B, $f = g$ if and only if $|\{f > t\}\Delta\{g > t\}| = 0$ for any $t > 0$. As $f \ne C_\omega f$ there exists $t > 0$ such that

$$
|\,\{f > t\}\,\Delta\,\{C_\omega f > t\}\,| > 0.
$$

By (4.3) $|\{f > t\}| = |\{C_\omega f > t\}|$ and it follows that $|\{f \le t < C_\omega f\}| = |\{C_\omega f \le t < f\}|$. Put $A := \{f \le t < C_\omega f\}$ and $A' := \{C_\omega f \le t < f\}$. For later use we note that

$$\mathcal{H}^{n-1}(A \cap \mathbb{S}^{n-1}_\tau) = \mathcal{H}^{n-1}(A' \cap \mathbb{S}^{n-1}_\tau)$$

for $\mathscr{L}^1$-a.e. $\tau \in (0, 1)$.

We claim there exists $H \in \mathscr{H}_\omega$ such that $|A \cap \tau_H A'| > 0$. Taking this as read, on $A \cap \tau_H A'$ we have that $C_\omega f > t \ge C_\omega f \circ \tau_H$ so that $A \cap \tau_H A' \subset H$. Also, $f \le t < f \circ \tau_H$ there. In short, $A \cap \tau_H A' \subset \{f \circ \tau_H > f\} \cap H$. So $|\{f \circ \tau_H > f\} \cap H| > 0$ and strict inequality holds by Lemma 4.4 (iii).

To prove the claim, assume for a contradiction that $|A \cap \tau_H A'| = 0$ for all $H \in \mathscr{H}_\omega$. Let F be a countable dense subset in $\mathbb{S}^{n-1} \cap H_\omega$. Then

$$\Big|\bigcup_{\nu \in F} (A \cap \tau_{H_\nu} A')\Big| = 0.$$

Therefore for all $r \in (0, 1)$ it holds that

$$\mathcal{H}^{n-1}(A_r \cap \tau_{H_\nu} A'_r) = 0 \quad \text{for every } \nu \in F,$$

except on a $\mathscr{L}^1$-null set $N \subset (0, 1)$. We write $A_r := A \cap \mathbb{S}^{n-1}_r$ for the r-section of A and likewise for A'. Let $\nu \in \mathbb{S}^{n-1} \cap H_\omega$ with corresponding reflection $\tau = \tau_{H_\nu}$. Select a sequence (ν_h) in F which converges to ν in $\mathbb{S}^{n-1}$ and write τ_h for the reflection associated to the closed half-space H_{ν_h}. For $r \in (0,\ 1) \setminus N$,

$$|\mathcal{H}^{n-1}(A_r \cap \tau A'_r) - \mathcal{H}^{n-1}(A_r \cap \tau_h A'_r)| \le \|\chi_{A'} - \chi_{A'} \circ \tau \circ \tau_h\|_{L^1(\mathbb{S}^{n-1}_r, \mathcal{H}^{n-1})},$$

and this latter converges to zero as $h \to \infty$ by Lemma 4.5. We derive that

$$\mathcal{H}^{n-1}(A_r \cap \tau_{H_\nu} A'_r) = 0 \text{ for every } \nu \in \mathbb{S}^{n-1} \cap H_\omega \tag{4.5}$$

for all $r \in (0,\ 1) \setminus N$.

To conclude the argument, choose $r \in (0, 1) \setminus N$ such that $\mathcal{H}^{n-1}(A_r) = \mathcal{H}^{n-1}(A'_r) > 0$. Select a density point x for A_r lying in A_r using [1, Corollary 2.23] for example; that is,

$$\lim_{\rho \downarrow 0} \frac{1}{\mathcal{H}^{n-1}(B(x, \rho)_r)} \int_{B(x,\rho)_r} |\chi_{A_r}(z) - \chi_{A_r}(x)|\, \mathcal{H}^{n-1}(dz) = 0.$$

This means that A_r has density 1 at x in the sense that

$$\frac{\mathcal{H}^{n-1}(A_r \cap B(x, \rho))}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1}_r \cap B(x, \rho))} \to 1 \text{ as } \rho \downarrow 0.$$

Choose y in A'_r similarly so that A'_r has density 1 at y. Then $C_\omega f(x) > t \geq C_\omega f(y)$. So there exists $\nu \in \mathbb{S}^{n-1} \cap H_\omega$ such that with $\tau = \tau_{H_\nu}$ we have that $\tau y = x$. But then

$$\lim_{\rho \downarrow 0} \frac{\mathcal{H}^{n-1}(A_r \cap \tau B_r \cap B(x,\rho))}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1} \cap B(x,\rho))} = 1,$$

so that, in fact, $\mathcal{H}^{n-1}(A_r \cap \tau B_r) > 0$, contradicting (4.5). □

Proof of Theorem 4.1 Let E be an optimal configuration for the data (U, p). Assume for a contradiction that $E \neq C_\omega E$ $\mathscr{L}^n$-a.e. for any $\omega \in \mathbb{S}^{n-1}$. Then there exists $\omega \in \mathbb{S}^{n-1}$ such that

$$\delta := \inf_{\nu \in \mathbb{S}^{n-1}} \|\chi_E - C_\nu \chi_E\|_{L^1(B)} = \|\chi_E - C_\omega \chi_E\|_{L^1(B)} > 0.$$

By Lemma 4.6 there exists $H \in \mathscr{H}_\omega$ such that $\|(\chi_E)_H - C_\omega \chi_E\|_{L^1(B)} < \|\chi_E - C_\omega \chi_E\|_{L^1(B)}$. It is plain from this that $\chi_E \neq (\chi_E)_H$; but also $\chi_E \circ \tau_H \neq (\chi_E)_H$, for otherwise,

$$\begin{aligned} \|\chi_E - C_{\tau_H \omega} \chi_E\|_{L^1(B)} &= \|\chi_E - C_\omega \chi_E \circ \tau_H\|_{L^1(B)} = \|(\chi_E)_H - C_\omega \chi_E\|_{L^1(B)} \\ &< \|\chi_E - C_\omega \chi_E\|_{L^1(B)}, \end{aligned}$$

contradicting optimality of ω. It follows by Theorem 4.2 that $J(E) < J(E_H)$, contradicting the fact that E is an optimal configuration for the data (U, p). The result now follows. □

5 Preliminaries on Weighted Dirichlet Forms

Let $n = 1$ and $U = (a, b)$ be an open bounded interval in $\mathbb{R}$. We are given a density function w with the property

(A) w is a positive function in $C_0(U)$.

The weighted volume of an $\mathscr{L}^1$-measurable set E in U is given by $m(E) := \int_E w\,dx$. We introduce the further assumption

(B) $w \in C^1(U)$ and $w'/w \in L^2(U, m)$.

Consider the coercive bilinear form

$$\mathscr{E}(u, v) := \int_U \left(uv + u'v'\right) dm \qquad (u, v \in \mathscr{D} := C^\infty(\overline{U}))$$

in $L^2(U, m)$.

Lemma 5.1 *Assume (A)–(B). Then*

(i) $(\mathscr{D}, \mathscr{E})$ *is closable in* $L^2(U, m)$ *with closure denoted* $(D(\mathscr{E}), \mathscr{E})$*;*
(ii) $(D(\mathscr{E}), \mathscr{E})$ *is a symmetric Dirichlet form in* $L^2(U, m)$.

Proof We refer to [19, Definitions I.2.3 and I.4.5] (for example). Note that $(\mathscr{D}, \mathscr{E})$ satisfies the weak sector condition [19] (2.3) by the Cauchy–Schwarz inequality. Suppose (u_h) is a sequence in $\mathscr{D}$ such that $u_h \to 0$ in $L^2(U, m)$. For $v \in \mathscr{D}$ an integration-by-parts gives

$$\begin{aligned}\int_U u_h' v' w\, dx &= \int_{\partial U} u_h v' w \nu \, d\mathcal{H}^0 - \int_U u_h (v'w)'\, dx \\ &= -\int_U u_h \frac{(v'w)'}{w}\, dm = -\int_U u_h \left\{ v'' + v' \frac{w'}{w} \right\} dm \to 0\end{aligned}$$

as $h \to \infty$ where $\nu = \pm 1$ is the one-dimensional unit exterior normal on ∂U. We have made use of the assumptions (A) and (B). The statement (i) follows by [19, Lemma I.3.4]. Then $(D(\mathscr{E}), \mathscr{E})$ is a symmetric closed form by definition (cf. [19, Definition I.2.3]). By [19, Proposition I.4.10 and II.2 (c)], $(D(\mathscr{E}), \mathscr{E})$ is a symmetric Dirichlet form. □

Given a real-valued function u on $\mathbb{R}_+$ (or $\mathbb{R}$) define the function $\theta_t u$ on $\mathbb{R}_+$ for each $t > 0$ by $(\theta_t u)(x) := u(x + t)$ for $x \in \mathbb{R}_+$ (or $\mathbb{R}$).

Lemma 5.2 *Let λ be a positive $\mathscr{L}^1$-integrable function on $\mathbb{R}_+$ such that*

$$c := \sup_{t>0} \sup_{x>0} \frac{\lambda(x)}{\lambda(x+t)} < \infty.$$

Then

(i) $\theta_t \in B(L^2(\mathbb{R}_+, \lambda\mathscr{L}^1))$ *for each* $t > 0$*;*
(ii) $\|u - \theta_t u\|_{L^2(\mathbb{R}_+, \lambda\mathscr{L}^1)} \to 0$ *as* $t \downarrow 0$ *for each* $u \in L^2(\mathbb{R}_+, \lambda\mathscr{L}^1)$.

Proof (i) For any $u \in L^2(\mathbb{R}_+, \lambda\mathscr{L}^1)$,

$$\begin{aligned}\|\theta_t u\|^2_{L^2(\mathbb{R}_+, \lambda\mathscr{L}^1)} &= \int_0^\infty |\theta_t u|^2 \lambda\, dx \le c \int_0^\infty |u(x+t)|^2 \lambda(x+t)\, dx \\ &\le c \|u\|^2_{L^2(\mathbb{R}_+, \lambda\mathscr{L}^1)}.\end{aligned}$$

(ii) The statement holds for $u \in C([0, \infty)) \cap L^2(\mathbb{R}_+, \lambda\mathscr{L}^1)$ by the dominated convergence theorem and this latter set is dense in $L^2(\mathbb{R}_+, \lambda\mathscr{L}^1)$. These observations as well as *(i)* lead to the result using a 3ε-argument. □

Our next assumption is stronger than required but easy to state:

(C) w is unimodal on U.

Lemma 5.3 *Assume (A)–(C). Then*

$$D(\mathscr{E}) = \left\{ u \in L^2(U, m) : u \textit{ is weakly differentiable on } U \textit{ and } u' \in L^2(U, m) \right\}.$$

Proof Let $u \in D(\mathscr{E})$. There exists a Cauchy sequence (u_h) in $(\mathscr{D}, \mathscr{E})$ which converges to u in $L^2(U, m)$. Then (u_h') is a Cauchy sequence in $L^2(U, m)$ with limit $v \in L^2(U, m)$ (say). For $\phi \in C_c^\infty(U)$,

$$\int_U u\phi'\,dx = \lim_h \int_U u_h\phi'\,dx = -\lim_h \int_U u_h'\phi\,dx = -\lim_h \int_U u_h'(w^{-1}\phi)\,dm$$
$$= -\int_U v\phi\,dx;$$

so u is weakly differentiable on U with weak derivative $u' = v \in L^2(U, m)$.

Now let $u \in L^2(U, m)$ be weakly differentiable on U such that $u' \in L^2(U, m)$. Multiplying by a partition of unity we may assume that $u = 0$ near b. Denote by $\overline{u}$ the extension of u to $\mathbb{R}$ by zero. For $t > 0$ put $v_t := (\theta_t\overline{u})|_U$. Note that v_t is weakly differentiable and $v_t' = \theta_t(\overline{u'})|_U$. For $t > 0$, $v_t, v_t' \in L^2(U, m)$. Let $(\rho_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers on $\mathbb{R}$ (cf. [1, 2.1]). For $t > 0$ and $\varepsilon > 0$ small, $(\rho_\varepsilon \star (\theta_t\overline{u}))' = \rho_\varepsilon \star (\theta_t\overline{u'})$ on U. The operation $\star$ stands for convolution. For $t > 0$ and $\varepsilon > 0$ small put $w_{t,\varepsilon} := \rho_\varepsilon \star (\theta_t\overline{u})|_U \in C^\infty(\overline{U})$. Now

$$\|u - w_{t,\varepsilon}\|_{L^2(U,m)} \leq \|u - v_t\|_{L^2(U,m)} + \|v_t - w_{t,\varepsilon}\|_{L^2(U,m)};$$
$$\|u' - w_{t,\varepsilon}'\|_{L^2(U,m)} \leq \|u' - v_t'\|_{L^2(U,m)} + \|v_t' - \rho_\varepsilon * (\theta_t\overline{u'})|_U\|_{L^2(U,m)}.$$

By Lemma 5.2 and (A)–(C) the expressions $\|u - v_t\|_{L^2(U,m)}$ and $\|u' - v_t'\|_{L^2(U,m)}$ are small for $t > 0$ small. We also use the fact that the mollified functions are regular approximations in L^2 (cf. [1, 2.1]). This shows that $u \in D(\mathscr{E})$. □

Suppose that $\hat{U} = (c, d)$ is an open bounded interval in $\mathbb{R}$ and $\Phi : \hat{U} \to U$ is a C^1 bijection such that $\Phi' =: \varphi > 0$ on $\hat{U}$. Let $\hat{m} := \Phi_\sharp m$ be the pull-back of m under Φ; thus $\hat{m} = \hat{w}\mathscr{L}^1$ on $\hat{U}$ where $\hat{w} := \varphi(w \circ \Phi)$. Define a coercive bilinear form

$$\hat{\mathscr{E}}(u, v) := \int_{\hat{U}} \left(uv + \varphi^{-2}u'v'\right) d\hat{m} \qquad (u, v \in D(\hat{\mathscr{E}}))$$

in $L^2(\hat{U}, \hat{m})$ with domain

$$D(\hat{\mathscr{E}}) := \left\{u \in L^2(\hat{U}, \hat{m}) : u \text{ is weakly differentiable on } \hat{U} \text{ and } \varphi^{-1}u' \in L^2(\hat{U}, \hat{m})\right\}.$$

Lemma 5.4 *Assume (A)–(C). Then*

(i) $(D(\hat{\mathscr{E}}), \hat{\mathscr{E}})$ *is a symmetric Dirichlet form in* $L^2(\hat{U}, \hat{m})$*;*
(ii) *the mapping* $D(\hat{\mathscr{E}}) \to D(\mathscr{E})$; $u \mapsto \overline{u} := u \circ \Phi^{-1}$ *is a Hilbert space isomorphism.*

Proof (i) We show that $(D(\hat{\mathscr{E}}), \hat{\mathscr{E}})$ is closed in $L^2(\hat{U}, \hat{m})$. Let (u_h) be an $\hat{\mathscr{E}}^{1/2}$-Cauchy sequence in $D(\hat{\mathscr{E}})$. Then (u_h) resp. $(\varphi^{-1}u_h')$ are Cauchy sequences in $L^2(\hat{U}, \hat{m})$ with limits $u \in L^2(\hat{U}, \hat{m})$ resp. $v \in L^2(\hat{U}, \hat{m})$. For $\phi \in C_c^\infty(\hat{U})$,

$$\int_{\hat{U}} u\phi' \, dx = \lim_{h\to\infty} \int_{\hat{U}} u_h \phi' \, dx = -\lim_{h\to\infty} \int_{\hat{U}} u_h' \phi \, dx$$
$$= -\lim_{h\to\infty} \int_{\hat{U}} \varphi^{-1} u_h' \frac{\phi}{w \circ \Phi} \, d\hat{m} = -\int_{\hat{U}} v \frac{\phi}{w \circ \Phi} \, d\hat{m} = -\int_{\hat{U}} (\varphi v)\phi \, dx$$

so u is weakly differentiable and $\varphi^{-1}u' = v$; that is, $u \in D(\hat{\mathscr{E}})$. It then follows that (u_h) converges to u in $\hat{\mathscr{E}}^{1/2}$-norm. So $(D(\hat{\mathscr{E}}), \hat{\mathscr{E}})$ is a symmetric closed form in $L^2(\hat{U}, \hat{m})$.

Let $u \in D(\hat{\mathscr{E}})$. Given $\varepsilon > 0$ let φ_ε be as in [19, Example II.2.7]. Note that $u \in W^{1,2}_{\mathrm{loc}}(\hat{U})$. Then $\varphi_\varepsilon(u)$ is weakly differentiable on $\hat{U}$ and $\varphi_\varepsilon(u)' = \varphi_\varepsilon'(u)u'$ (see for example [10, 5.10, Exercise 16]) so $\varphi_\varepsilon(u) \in D(\hat{\mathscr{E}})$. We then derive that

$$\hat{\mathscr{E}}(\varphi_\varepsilon(u), \varphi_\varepsilon(u)) \leq \hat{\mathscr{E}}(u, u).$$

By [19, Proposition I.4.7], $(D(\hat{\mathscr{E}}), \hat{\mathscr{E}})$ is a symmetric Dirichlet form in $L^2(\hat{U}, \hat{m})$.

(ii) Let $u \in D(\hat{\mathscr{E}})$. Note that $(u'/\varphi) \circ \Phi^{-1} \in L^2(U, m)$ (use [1, (2.47)]) and $\overline{u}$ is weakly differentiable on U with $\overline{u}' = (u'/\varphi) \circ \Phi^{-1} \in L^2(U, m)$. Thus the mapping is well-defined. For $u, v \in D(\hat{\mathscr{E}})$, $\hat{\mathscr{E}}(u, v) = \mathscr{E}(\overline{u}, \overline{v})$ again using [1, (2.47)]. In particular, the mapping $u \mapsto \overline{u}$ is injective. Now let $u \in D(\mathscr{E})$ and put $\hat{u} := u \circ \Phi$. Then $\hat{u} \in L^2(\hat{U}, \hat{m})$, $\hat{u}$ is weakly differentiable on $\hat{U}$ with weak derivative $\hat{u}' = \varphi(u' \circ \Phi)$ and $\varphi^{-1}\hat{u}' \in L^2(\hat{U}, \hat{m})$; in other words, $\hat{u} \in D(\hat{\mathscr{E}})$. This shows that the mapping in (ii) is surjective. □

Now take $B = (-1, 1)$. We are given density functions f, g with the properties

(A.1) f is a positive function in $C(B)$;
(A.2) g is a positive unimodal function in $C_0(B)$.

The weighted volume of an $\mathscr{L}^1$-measurable set E in B is given by $V(E) := \int_E f \, dx$. Put $\rho := f/g$. We introduce the further assumption

(A.3) $\rho \in L^1(B, \mathscr{L}^1)$, $g \in C^1(B)$ and $g'/f \in L^2(B, V)$.

Define

$$R : B \to \mathbb{R}; t \mapsto \int_0^t \rho(\tau) \, d\tau,$$

and let $\check{B}$ denote the image of B under R; $\check{B}$ is an open bounded interval in $\mathbb{R}$. Then $R : B \to \check{B}$ is a C^1 bijection. Define $\check{g} := g \circ R^{-1}$ on $\check{B}$. Define the measure $\check{V} := \check{g}\mathscr{L}^1$ on $\check{B}$.

We introduce coercive bilinear forms

$$\check{\mathscr{E}}(u, v) := \int_{\check{B}} \Big(uv + u'v'\Big) \, d\check{V} \qquad (u,\ v \in D(\check{\mathscr{E}}))$$

in $L^2(\check{B}, \check{V})$ with domain

$$D(\check{\mathscr{E}}) = \Big\{u \in L^2(\check{B}, \check{V}) : u \text{ is weakly differentiable on } \check{B} \text{ and } u' \in L^2(\check{B}, \check{V})\Big\};$$

and

$$\hat{\mathscr{E}}(u, v) := \int_B \Big(uv + \rho^{-2} u'v'\Big)\, dV \qquad (u,\ v \in D(\hat{\mathscr{E}}))$$

in $L^2(B, V)$ with domain

$$D(\hat{\mathscr{E}}) = \Big\{u \in L^2(B, V) : u \text{ is weakly differentiable on } B \text{ and } \rho^{-1}u' \in L^2(B, V)\Big\}.$$

Note that with $\check{B}$ in place of U, $w := \check{g}$ satisfies properties (A)–(C) above in light of the assumptions (A.1)–(A.3). We derive

Lemma 5.5 *Assume (A.1)–(A.3). Then*

(i) $(D(\check{\mathscr{E}}), \check{\mathscr{E}})$ *is a symmetric Dirichlet form in* $L^2(\check{B}, \check{V})$*;*
(ii) $C^\infty(\overline{\check{B}})$ *is dense in* $D(\check{\mathscr{E}})$ *with respect to the* $\check{\mathscr{E}}^{1/2}$*-norm;*
(iii) $(D(\hat{\mathscr{E}}), \hat{\mathscr{E}})$ *is a symmetric Dirichlet form in* $L^2(B, V)$*;*
(iv) *the mapping* $D(\hat{\mathscr{E}}) \to D(\check{\mathscr{E}})$; $u \mapsto u \circ R^{-1}$ *is a Hilbert space isomorphism.*

6 An (f, g)-Isoperimetric Inequality

Recall that an $\mathscr{L}^1$-measurable set $E \subset B$ is said to be a Caccioppoli set if for each relatively compact open set Ω in B,

$$P(E, \Omega) := \sup\left\{\int_\Omega \chi_E \phi'\, dx : \phi \in C_c^\infty(\Omega, \mathbb{R}), \|\phi\|_\infty \leq 1\right\} < \infty.$$

There then exists a unique real Radon measure $D\chi_E$ on B such that

$$\int_B \chi_E \phi'\, dx = -\int_B \phi\, dD\chi_E$$

for all $\phi \in C_c^\infty(B, \mathbb{R})$ [1, Corollary 1.55]. Denote by $|D\chi_E|$ the total variation measure of $D\chi_E$.

Theorem 6.1 *Suppose that* E *is a Caccioppoli set in* B *with* $|E| > 0$*. Then there exist* $N \in \mathbb{N} \cup \{\infty\}$ *and closed intervals* $E_h = [a_{2h-1}, a_{2h}] \subset \mathbb{R}$ $(h = 1, \ldots, N)$ *with non-empty interior and separated by open neighbourhoods in* $\mathbb{R}$ *such that* E *is* $\mathscr{L}^1$*-a.e. equivalent to the union of the* E_h*.*

The statement that the collection of intervals (E_h) is separated by open intervals means that $\inf_{k \neq h} d(E_h, E_k) > 0$ for each h. Here, d denotes the standard metric on $\mathbb{R}$.

Proof The proof is along the lines of [1, Proposition 3.52]. Put $u := \chi_E \in \mathrm{BV}_{\mathrm{loc}}(B)$. Then $\mu := Du$ is a real Radon measure on B [1, 1.40]. Define

$$w(t) := \begin{cases} -\mu([t,0)) & \text{for } -1 < t < 0; \\ 0 & \text{for } t = 0; \\ \mu([0,t)) & \text{for } 0 < t < 1; \end{cases}$$

w is left-continuous on B by the Hahn decomposition and inner/outer regularity [1, 1.43]. By Fubini's theorem, $w \in \mathrm{BV}_{\mathrm{loc}}(B)$ and $Dw = \mu$. By [1, Proposition 3.2], $u = c + w$ $\mathscr{L}^1$-a.e. on B for some $c \in \mathbb{R}$. Let A be the set of atoms of μ in B. Note that w is continuous on $B \setminus A$ and $w(t+) - w(t) = \mu(\{t\})$ at each $t \in A$. As $c + w \in \{0, 1\}$ $\mathscr{L}^1$-a.e. on B, $\mu(\{t\}) \in \{-1, 1\}$ for each $t \in A$. Let Ω be a relatively compact open set in B. Then $\mathrm{Card}(A \cap \Omega) = |\mu|(A \cap \Omega) \leq |\mu|(\Omega) < \infty$. Thus the set of atoms A accumulates at ∂B (if at all). By the observations above, the function $c + w$ is constant on each connected component of $B \setminus A$ with values in the set $\{0, 1\}$. Let the sets E_h be the closure of the open intervals in $B \setminus A$ where $c + w$ takes the value 1. □

Let g be a positive lower semicontinuous function on B. Let E be a Caccioppoli set in B. The g-perimeter of E relative to B is defined by

$$P_g(E, B) := \int_B g \, d|D\chi_E|. \tag{6.1}$$

Lemma 6.2 *Let g be a positive lower semicontinuous function on B and E a Caccioppoli set in B. Then*

$$P_g(E, B) = \sum_{a \in A} g(a)$$

where A stands for the set of atoms of $D\chi_E$ in B.

Proof A direct computation gives $D\chi_E = \sum_{a \in A} D\chi_E(\{a\})\delta_a$ and $|D\chi_E| = \sum_{a \in A} \delta_a$ which gives the result. We use δ_a to stand for the Dirac measure at a. □

Let f, g be densities on B satisfying conditions (A.1)–(A.3). The weighted volume of an $\mathscr{L}^1$-measurable set E in B is the measure given by $V(E) := \int_E f \, dx$. Define $F : [-1, 1] \to [0, V(B)]$ by $F(x) := V((-1, x))$ and

$$J(p) := (g \circ F^{-1})(p) \text{ for } p \in [0, V(B)].$$

We impose the additional assumptions

(A.4) $J(p) = J(V(B) - p)$ for $0 < p < V(B)$;
(A.5) for all $p, q > 0$ with $p + q < V(B)$, $J(p + q) < J(p) + J(q)$.

For an $\mathscr{L}^1$-measurable set E in B the $\star$-rearrangement of E is defined by $E^\star := (-1, F^{-1}(V(E)))$. We then have that the following (f, g)-isoperimetric inequality is valid.

Theorem 6.3 *Assume (A.1)–(A.5). Suppose that E is a Caccioppoli set in B. Then $P_g(E, B) \geq P_g(E^\bigstar, B)$.*

Proof For $0 < p < q < V(B)$,

$$J(p) + J(q) = J(p) + J(V(B) - q) > J(p + V(B) - q) = J(q - p) \quad (6.2)$$

by (A.4)–(A.5). The above inequality also holds for $0 \leq p < q \leq V(B)$ with $>$ replaced by $\geq$ by continuity; equality holds if and only if one or both of p, q are extremal.

We may suppose that $V(E) = p$ for some $p \in (0, V(B))$. Assume that E has the form $E = \bigcup_{h=1}^{N}[a_{2h-1}, a_{2h}]$ with $-1 \leq a_1 < a_2 < \cdots < a_{2N-1} < a_{2N} \leq 1$ for some $N \in \mathbb{N}$. Put $p_j := F(a_j)$ so that $0 \leq p_1 < \cdots < p_{2N} \leq V(B)$ and $\sum_{h=1}^{N}(p_{2h} - p_{2h-1}) = p$. By Lemma 6.2, (6.2) and (A.5),

$$\begin{aligned} P_g(E, B) &= \sum_{h=1}^{N}\Big(J(p_{2h-1}) + J(p_{2h})\Big) \geq \sum_{h=1}^{N} J(p_{2h} - p_{2h-1}) \\ &\geq J\left(\sum_{h=1}^{N}(p_{2h} - p_{2h-1})\right) = J(p) = P_g(E^\bigstar, B). \end{aligned}$$

The result for arbitrary E as in the statement follows by Theorem 6.1, the monotone convergence theorem and continuity of J. □

We now investigate the equality case in the isoperimetric inequality.

Theorem 6.4 *Assume (A.1)–(A.5). Suppose that E is a Caccioppoli set in B. Assume that $P_g(E, B) = P_g(E^\bigstar, B)$. Then either $E = E^\bigstar$ or $B \setminus E = (B \setminus E)^\bigstar$ V-a.e.*

Proof We may assume that E is the union of closed intervals $E_h \subset \mathbb{R}$ ($h = 1, \ldots, N, N \in \mathbb{N} \cup \{\infty\}$) with non-empty interior and separated by open neighbourhoods in $\mathbb{R}$ as in Theorem 6.1. In virtue of (6.1) we may take $N > 1$. We can then find $x \in B \setminus E$ such that $V(E \cap (x, 1)) > 0$ and $V(E \cap (-1, x)) > 0$. Put $E_- := E \cap (-1, x)$, $E_+ := E \cap (x, 1)$, $F_- := E_-^\bigstar$ and $F_+ := B \setminus (B \setminus E_+)^\bigstar$. Note that $V(E_-) = V(F_-) = p_-$ and $V(E_+) = V(F_+) = p_+$ for some $p_\pm \in (0, V(B))$. We have

$$\begin{aligned} J(p_-) + J(p_+) &= P_g(F_-, B) + P_g(F_+, B) \leq P_g(E_-, B) + P_g(E_+, B) \\ &= P_g(E, B) \end{aligned}$$

by Theorem 6.3. On the other hand, $P_g(E, B) = P_g(E^\bigstar, B) = J(p_- + p_+) < J(p_-) + J(p_+)$ by (A.5), a contradiction. □

Proposition 6.5 *Assume (A.1)–(A.3). If g'/f is strictly decreasing on B then (A.5) holds.*

Proof Note that J is differentiable on $(0, V(B))$. Moreover, $(F^{-1})'(p) = 1/(f \circ F^{-1})(p)$ and

$$J'(p) = g'(F^{-1}(p)) \cdot 1/(f \circ F^{-1})(p) = (g'/f)(F^{-1}(p))$$

for $p \in (0, V(B))$; this shows that J is strictly concave on $(0, V(B))$ as $(0, V(B)) \mapsto (-1, 1) : p \mapsto (F^{-1})(p)$ is increasing. For $0 < p < q < V(B)$ we have that

$$\frac{J(p)}{p} \geq \frac{J(q)}{q} \tag{6.3}$$

by concavity and the fact that $J(0) \geq 0$. Supposing additionally that $p + q < V(B)$ we have

$$\frac{J(q) - J(p)}{q - p} > \frac{J(p+q) - J(q)}{p}$$

by considering the gradient of chords. Rearranging and using (6.3) gives (A.5). □

Let u be a real-valued $\mathscr{L}^1$-measurable function on B. Put $\mu_u(t) := V(\{|u| > t\})$ for $t \geq 0$. The function $\mu_u : [0, \infty) \to [0, V(B)]$ is non-increasing, right-continuous and $\mu_u(t) \to 0$ as $t \to \infty$. Define its right-continuous inverse $u^\sharp : [0, V(B)] \to [0, \infty]$ by

$$u^\sharp(s) := \inf\{t \geq 0 : \mu_u(t) \leq s\} \text{ for } 0 \leq s \leq V(B),$$

with the understanding that $\inf \emptyset = +\infty$. Define $u^\bigstar := u^\sharp \circ F$ on B. Note that $\mu_u(t) > s$ if and only if $u^\sharp(s) > t$ (see Lemma 10.1). It follows that

$$V(\{|u| > t\}) = V(\{u^\bigstar > t\}) \text{ for each } t \geq 0; \tag{6.4}$$

in fact, $V(\{u^\bigstar > t\}) = V(\{F < \mu_u(t)\}) = V((-1, F^{-1}(\mu_u(t)))) = \mu_u(t)$.
The following result is a Hardy–Littlewood type inequality and can be proved as in [14, 13.10] (see also [9, Theorem 3]).

Theorem 6.6 *Assume (A.1)–(A.5). Let u, v be real-valued $\mathscr{L}^1$-measurable functions on B. Then*

$$\int_B |uv| \, dV \leq \int_B u^\bigstar v^\bigstar \, dV.$$

The next non-expansivity result can be found in [9, Corollary 1].

Theorem 6.7 *Assume (A.1)–(A.5). Let u, v be real-valued $\mathscr{L}^1$-measurable functions on B. Then*

$$\int_B |u^\bigstar - v^\bigstar|^2 \, dV \leq \int_B |u - v|^2 \, dV.$$

Define a metric $\hat{d}$ on B as follows. The length of a piecewise C^1 parametrised curve $\gamma : [\alpha, \beta] \to B$ in $(B, \hat{d})$ is

$$\hat{L}[\gamma] = \int_\alpha^\beta \rho(\gamma(t))|\dot{\gamma}(t)|\, dt.$$

For $x, y \in B$, $\hat{d}(x, y)$ stands for the infimum of lengths of piecewise C^1 parametrised curves in B connecting x to y; $\hat{d}(\cdot, \cdot)$ is a metric on B.

Lemma 6.8 *For $x, y \in B$, $\hat{d}(x, y) = d(R(x), R(y))$.*

Proof Let $\gamma : [\alpha, \beta] \to B$ be a piecewise C^1 parametrised curve in B connecting x to y. Then $\hat{L}[\gamma] = L[R \circ \gamma]$ in an obvious notation. So $\hat{d}(x, y) \le d(R(x), R(y))$. A similar argument gives the reverse inequality. □

Note that for each $\mathscr{L}^1$-measurable set E in B, $\check{V}(R(E)) = V(E)$.

Lemma 6.9 *The mapping $R : B \to \check{B}$ sets up a one-to-one correspondence between Caccioppoli sets in B resp. $\check{B}$. Moreover, let E be a Caccioppoli set in B. Then*

(i) $|D\chi_{R(E)}| = R_\sharp |D\chi_E|$;
(ii) $P_{\check{g}}(R(E), \check{B}) = P_g(E, B)$.

Proof Let $\Omega \subset B$ be a relatively compact open set. Then

$$\begin{aligned} |D\chi_{R(E)}|(R(\Omega)) &= \sup \left\{ \int_{R(\Omega)} \chi_{R(E)} \phi'\, dx : \phi \in C_c^1(\check{B}, \mathbb{R}) \text{ and } \|\phi\|_\infty \le 1 \right\} \\ &= \sup \left\{ \int_\Omega \chi_E (\phi \circ R)'\, dx : \phi \in C_c^1(\check{B}, \mathbb{R}) \text{ and } \|\phi\|_\infty \le 1 \right\} \\ &= \sup \left\{ \int_\Omega \chi_E \phi'\, dx : \phi \in C_c^1(B, \mathbb{R}) \text{ and } \|\phi\|_\infty \le 1 \right\} \\ &= |D\chi_E|(\Omega), \end{aligned}$$

from which the first assertion follows. Item (i) follows from the definition of the push-forward [1, Definition 1.7] and the coincidence criterion [1, Proposition 1.8], while (ii) follows from (6.1), (i) and the change of variables formula for integrals. □

The function $\check{F} : \mathbb{R} \to [0, V(B)]$ defined by $\check{F}(x) := \check{V}(\check{B} \cap (-\infty, x))$ is the cumulative distribution function of $\check{g}$. Let u be a real-valued $\mathscr{L}^1$-measurable function on $\check{B}$. Put $\check{\mu}_u(t) := \check{V}(\{|u| > t\})$ for $t \ge 0$ and denote by $\check{u}^\sharp : [0, \check{V}(\check{B})] \to [0, \infty]$ its right-continuous inverse (as in the Appendix). Define $u^\star := \check{u}^\sharp \circ \check{F}$ on $\check{B}$.

Proposition 6.10 *Let u be a real-valued $\mathscr{L}^1$-measurable function on B and put $v := u \circ R^{-1}$. Then $u^\bigstar = v^\star \circ R$. In particular, for any $\mathscr{L}^1$-measurable set $E \subset B$, $R(E)^\star = R(E^\bigstar)$.*

Proof We have that $\mu_u(t) = \check{\mu}_v(t)$ for each $t \ge 0$; hence $u^\sharp = \check{v}^\sharp$ on $[0, V(B)]$. Now , $\check{F} \circ R = F$ on $[-1, 1]$. This leads to the first claim. The second then follows straightforwardly. □

Corollary 6.11 *Assume (A.1)–(A.5). Suppose that E is a Caccioppoli set in $\check{B}$. Then $P_{\check{g}}(E, \check{B}) \geq P_{\check{g}}(E^\star, \check{B})$.*

Proof Let E be a Caccioppoli set in B. By Lemma 6.9, Theorem 6.3 and Proposition 6.10,

$$P_{\check{g}}(R(E), \check{B}) = P_g(E, B) \geq P_g(E^\bigstar, B) = P_{\check{g}}(R(E^\bigstar), \check{B}) = P_{\check{g}}(R(E)^\star, \check{B}).$$

□

Corollary 6.12 *Assume (A.1)–(A.5). Suppose that E is a Caccioppoli set in $\check{B}$. Assume that $P_{\check{g}}(E, \check{B}) = P_{\check{g}}(E^\star, \check{B})$. Then either $E = E^\star$ or $\check{B} \setminus E = (\check{B} \setminus E)^\star$ $\check{V}$-a.e.*

Proof This follows from Lemma 6.9 and Theorem 6.4. □

Finally, we state a counterpart of Theorem 6.7.

Theorem 6.13 *Assume (A.1)–(A.5). Let u, v be real-valued $\mathscr{L}^1$-measurable functions on $\check{B}$. Then*

$$\int_{\check{B}} |u^\star - v^\star|^2 \, d\check{V} \leq \int_{\check{B}} |u - v|^2 \, d\check{V}.$$

7 A Pólya–Szegö Inequality

We first show that the rearrangement $\cdot^\star$ is smoothing in the sense of [21] (see also [4]). Given $r > 0$ write E_r for the r-neighbourhood of an $\mathscr{L}^1$-measurable set E in $(\check{B}, d)$; by convention, $\emptyset_r = \emptyset$. The Minkowski content of E is the quantity

$$\check{V}^+(E) := \liminf_{r \downarrow 0} \frac{\check{V}(E_r) - \check{V}(E)}{r} \in [0, \infty].$$

Lemma 7.1 *Let E be a finite union of open intervals in $(\check{B}, d)$. Then*

(i) *E is a Caccioppoli set in $\check{B}$;*
(ii) *$\check{V}^+(E) = P_{\check{g}}(E, \check{B})$.*

Proof (i) The set E is a finite union of disjoint open intervals, $\overline{E}$ is a finite union of closed intervals in $\mathbb{R}$ with non-empty interior and separated by open sets in the sense of Theorem 6.1 and $\overline{E} = E \cup I$ for a finite set $I \subset \mathbb{R}$. So E is $\mathscr{L}^1$-a.e. equivalent to $F := \overline{E} \cap \check{B}$; in particular, E is a Caccioppoli set in $\check{B}$ by Theorem 6.1.
(ii) Suppose first that $F = [a_1, a_2] \subset \check{B}$. Then $E_r = F_r$ for each $r > 0$ and $\check{V}(E) = \check{V}(F)$ as $\check{V}$ is non-atomic. For small $r > 0$,

$$\begin{aligned} \frac{1}{r}\check{V}(E_r \setminus E) = \frac{1}{r}\check{V}(F_r \setminus F) &= \frac{1}{r}\int_{a_1 - r}^{a_1} \check{g} \, dy + \frac{1}{r}\int_{a_2}^{a_2 + r} \check{g} \, dy \\ &\to \int_{\partial F} \check{g} \, d\mathcal{H}^0 = P_{\check{g}}(F, \check{B}) = P_{\check{g}}(E, \check{B}) \end{aligned}$$

as $r \downarrow 0$. The result for general E as in the statement follows from the property that the closed intervals in $\overline{E}$ are separated by open sets. $\square$

For $p \in (0, V(B))$ we may write

$$\check{F}^{-1}(p) = \int_{V(B)/2}^{p} \frac{d\tau}{\check{J}(\tau)}$$

where $\check{J} = \check{g} \circ \check{F}^{-1}$. Note that $\check{J} = J$ due to the fact that $\check{F} \circ R = F$ on B.

Lemma 7.2 *Let E be an $\mathscr{L}^1$-measurable set in $\check{B}$. Then $\check{V}(E_r) \geq \check{V}((E^\star)_r)$ for each $r > 0$. In particular, the rearrangement $\cdot^\star$ is smoothing in the sense that $(E^\star)_r \subset (E_r)^\star$ for each $\mathscr{L}^1$-measurable set E in $\check{B}$ and $r > 0$.*

Proof We verify the conditions in [3, Theorem 2.1 (c)]. The measure $\check{V}$ is a separable non-atomic Borel measure on the metric space $(\check{B}, d)$. The r-neighbourhood ($r > 0$) of any open ball in $\check{B}$ is an open ball in $\check{B}$. Let E be a finite union of open intervals in $(\check{B}, d)$. By Lemma 7.1 and Corollary 6.11,

$$\check{V}^+(E) = P_{\check{g}}(E, \check{B}) \geq P_{\check{g}}(E^\star, \check{B}) = \check{J}(\check{V}(E)).$$

Thus by [3, Theorem 2.1], $\check{V}(E_r) \geq \check{F}(\check{F}^{-1}(\check{V}(E))+r) = \check{V}((E^\star)_r)$ for any Borel set E in $\check{B}$ with $0 < \check{V}(E) < V(B)$ and $r > 0$. The result then extends to $\mathscr{L}^1$-measurable sets in $\check{B}$. $\square$

Lemma 7.3 *Let A, E be $\mathscr{L}^1$-measurable sets in $\check{B}$ with $A \subset E$. Then $d(A^\star, \check{B} \setminus E^\star) \geq d(A, \check{B} \setminus E)$.*

Here, $d(A, E) := \inf\{d(x, y) : x \in A, y \in E\}$ with the understanding that $\inf \emptyset = +\infty$.

Proof We use the criterion that for $r > 0$, $A_r \subset E$ if and only if $d(A, \check{B} \setminus E) \geq r$. Put $r := d(A, \check{B} \setminus E)$; we may assume that $r > 0$. By the criterion, $A_r \subset E$ and hence $(A_r)^\star \subset E^\star$. By Lemma 7.2, $\check{V}((A_r)^\star) = \check{V}(A_r) \geq \check{V}((A^\star)_r)$ meaning $(A^\star)_r \subset (A_r)^\star \subset E^\star$ which entails that $d(A^\star, \check{B} \setminus E^\star) \geq r$ by the criterion. $\square$

The modulus of continuity of an arbitrary real-valued function u on $\check{B}$ is defined by

$$\omega_u(t) := \sup\left\{|u(x) - u(y)| : x,\ y \in \check{B} \text{ and } d(x, y) < t\right\} \in [0, \infty] \text{ for } t > 0.$$

Observe that u is uniformly continuous on $\check{B}$ if and only if $\lim_{t \downarrow 0} \omega_u(t) = 0$. We state the following criterion without proof.

Lemma 7.4 *Let u be a real-valued function on $\check{B}$ and $t, \tau > 0$. Then $\omega_u(t) > \tau$ if and only if there exist $s, s' \in \mathbb{R}$ with $s > s' + \tau$ such that $d(\{u > s\}, \check{B} \setminus \{u > s'\}) < t$.*

Proposition 7.5 *Assume (A.1)–(A.5).*

(i) *Let u be a real-valued $\mathscr{L}^1$-measurable function on $\check{B}$. Then $\omega_u(t) \geq \omega_{u^\star}(t)$ for each $t > 0$.*
(ii) *If u is uniformly continuous on $\check{B}$ then so is $u^\star$.*
(iii) *If u is Lipschitz continuous on $\check{B}$ then so is $u^\star$ and* $\mathrm{Lip}(u^\star, \check{B}) \leq \mathrm{Lip}(u, \check{B})$.

Proof Let $t > 0$. We may assume that $\omega_{u^\star}(t) > 0$. Choose $\tau > 0$ such that $\omega_{u^\star}(t) > \tau$. By Lemma 7.4 there exist $s, s' \in \mathbb{R}$ with $s > s' + \tau$ such that $d(\{u^\star > s\}, \check{B} \setminus \{u^\star > s'\}) < t$. Now $\{u^\star > s\} = \{|u| > s\}^\star$ and likewise for s' by the counterpart of the equimeasurabilty property (6.4). By Lemma 7.3 we deduce that $d(\{|u| > s\}, \check{B} \setminus \{|u| > s'\}) < t$ and again by Lemma 7.4 that $\omega_u(t) \geq \omega_{|u|}(t) > \tau$. Item (i) then follows. Part (ii) is a ready consequence. As for (iii),

$$\begin{aligned}\mathrm{Lip}(u, \check{B}) &:= \sup\left\{\frac{|u(x) - u(y)|}{d(x, y)} : x, y \in \check{B}, x \neq y\right\}\\ &= \sup_{t>0}(1/t)\omega_u(t) \geq \sup_{t>0}(1/t)\omega_{u^\star}(t) = \mathrm{Lip}(u^\star, \check{B}).\end{aligned}$$

□

Let u be a Lipschitz continuous function on $(\check{B}, d)$. By Rademacher's theorem (cf. [1, Theorem 2.14]) u is differentiable $\mathscr{L}^1$-a.e. on $\check{B}$ and its derivative coincides with the weak derivative on a set of full measure. Put

$$\begin{aligned}Z_1 &:= \left\{x \in \check{B} : u \text{ is differentiable at } x \text{ and } u'(x) = 0\right\},\\ Z_2 &:= \left\{x \in \check{B} : u \text{ is not differentiable at } x\right\} \text{ and } Z := Z_1 \cup Z_2.\end{aligned}$$

By [1, Lemmas 2.95 and 2.96], $Z \cap \{u = t\} = \emptyset$ for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$ and hence $N := u(Z) \subset \mathbb{R}$ is $\mathscr{L}^1$-negligible. The analogous sets corresponding to $u^\star$ will be decorated with the subscript $\star$.
We shall make use of the coarea formula [1, Theorem 2.93 and (2.74)],

$$\int_{\check{B}} \phi|u'|\,dx = \int_{-\infty}^{\infty}\int_{\check{B}\cap\{u=t\}} \phi\, d\mathcal{H}^0\, dt \tag{7.1}$$

for any $\mathscr{L}^1$-measurable function $\phi : \check{B} \to [0, \infty]$.

Lemma 7.6 *Let u be a nonnegative Lipschitz continuous function on $(\check{B}, d)$. Then*

(i) $\check{\mu}_u \in \mathrm{BV}(\mathbb{R})$;
(ii) $D\check{\mu}_u = -u_\sharp \check{V}$;
(iii) $D\check{\mu}_u^a = D\check{\mu}_u \llcorner (\mathbb{R} \setminus N)$;
(iv) $D\check{\mu}_u^s = D\check{\mu}_u \llcorner N$;
(v) $A := \left\{t \in \mathbb{R} : \mathscr{L}^1(Z \cap \{u = t\}) > 0\right\}$ *is the set of atoms of* $D\check{\mu}_u$ *and* $D\check{\mu}_u^j = D\check{\mu}_u \llcorner A$;

(vi) $\check{\mu}_u$ *is differentiable* $\mathscr{L}^1$*-a.e. on* $\mathbb{R}$ *with derivative given by*

$$\check{\mu}_u'(t) = -\int_{(\check{B}\setminus Z)\cap\{u=t\}} \frac{\check{g}}{|u'|}\, d\mathcal{H}^0$$

for $\mathscr{L}^1$*-a.e.* $t \in \mathbb{R}$*;*
(vii) $\mathrm{Ran}(u) = \mathrm{supp}(D\check{\mu}_u)$.

The notation above $D\check{\mu}_u^a$, $D\check{\mu}_u^s$, $D\check{\mu}_u^j$ stands for the absolutely continuous resp. singular resp. jump part of the measure $D\check{\mu}_u$ (see [1, 3.2] for example).

Proof For any $\varphi \in C_c^\infty(\mathbb{R})$,

$$\int_{-\infty}^{\infty} \check{\mu}_u \varphi'\, dt = \int_{\check{B}} \varphi \circ u\, d\check{V}$$

by Fubini's theorem; so $\check{\mu}_u \in \mathrm{BV}(\mathbb{R})$ and $D\check{\mu}_u$ is the push-forward of $\check{V}$ under u, $D\check{\mu}_u = -u_\sharp \check{V}$ (cf. [1, 1.70]). By (7.1),

$$D\check{\mu}_u \llcorner (\mathbb{R}\setminus N)(A) = -\check{V}(\{u \in A\} \cap \check{B} \setminus Z) = -\int_A \int_{(\check{B}\setminus Z)\cap\{u=t\}} \frac{\check{g}}{|u'|}\, d\mathcal{H}^0\, dt$$

for any $\mathscr{L}^1$-measurable set A in $\mathbb{R}$. In light of the above, we may identify $D\check{\mu}_u^a = D\check{\mu}_u \llcorner (\mathbb{R}\setminus N)$ and $D\check{\mu}_u^s = D\check{\mu}_u \llcorner N$. The set of atoms of $D\check{\mu}_u$ is defined by $A := \{t \in \mathbb{R} : D\check{\mu}_u(\{t\}) \neq 0\}$. By [13, Lemma 7.7], we may write A as in (v). The monotone function $\check{\mu}_u$ is a good representative within its equivalence class and is differentiable $\mathscr{L}^1$-a.e. on $\mathbb{R}$ with derivative given by the density of $D\check{\mu}_u$ with respect to $\mathscr{L}^1$ by [1, Theorem 3.28]. Item (vii) follows from (ii). □

Lemma 7.7 *Let u be a nonnegative Lipschitz continuous function on $(\check{B}, d)$. Then* $\int_{\check{B}\cap Z} u^2\, d\check{V} = \int_{\check{B}\cap Z_\star} (u^\star)^2\, d\check{V}$.

Proof As Z has finite $\mathscr{L}^1$-measure, $A \subset \mathbb{R}$ is a countable set. Thus

$$\int_{\check{B}\cap Z} u^2\, d\check{V} = \sum_{t\in A} t^2 \check{V}(Z \cap \{u = t\})) = \int_A t^2 d(u_\sharp \check{V}) = -\int_A t^2 d D\check{\mu}_u$$

and an analogous result holds for $u^\star$ by Lemma 7.6. The fact that $\check{\mu}_u = \check{\mu}_{u^\star}$ entails that $A = A_\star$. This leads to the result. □

Theorem 7.8 *Assume (A.1)–(A.5). Let u be a Lipschitz continuous function on $(\check{B}, d)$. Then $u, u^\star \in D(\check{\mathscr{E}})$ and $\check{\mathscr{E}}(u, u) \geq \check{\mathscr{E}}(u^\star, u^\star)$.*

Proof Given a Lipschitz continuous function u on $(\check{B}, d)$, $u \in W^{1,\infty}(\check{B})$ and $\|u'\|_{L^\infty(\check{B})} = \mathrm{Lip}(u, \check{B})$ (see [1, Proposition 2.13]) so $u \in D(\check{\mathscr{E}})$. The same is true for $u^\star$ by Proposition 7.5. Replacing u by $|u|$ and using the contraction property of the Dirichlet form $(D(\check{\mathscr{E}}), \check{\mathscr{E}})$ we may assume that u is nonnegative.

The proof hinges on the identity

$$\int_{\check{B}\setminus Z}\left(u^2+|u'|^2\right)d\check{V}=\int_{\mathrm{Ran}(u)}\int_{(\check{B}\setminus Z)\cap\{u=t\}}\frac{u^2+|u'|^2}{|u'|}\check{g}\,d\mathcal{H}^0\,dt, \tag{7.2}$$

which is a consequence of the coarea formula (7.1).
As $\check{\mu}_u=\check{\mu}_{u^\star}$ we have that $D\check{\mu}_u=D\check{\mu}_{u^\star}$. In particular, we derive that

$$\check{\mu}_u'(t)=-\int_{(\check{B}\setminus Z)\cap\{u=t\}}\frac{\check{g}}{|u'|}\,d\mathcal{H}^0=-\int_{(\check{B}\setminus Z_\star)\cap\{u^\star=t\}}\frac{\check{g}}{|u^{\star\prime}|}\,d\mathcal{H}^0=\check{\mu}_{u^\star}'(t) \tag{7.3}$$

for $\mathscr{L}^1$-a.e. $t\in\mathbb{R}$ by Lemma 7.6.
Let $t\in\mathrm{Ran}(u)$ be such that $Z\cap\{u=t\}=\emptyset$ and $\int_{(\check{B}\setminus Z)\cap\{u=t\}}\check{g}\,d\mathcal{H}^0<\infty$ and the analogous properties hold for $u^\star$. We assume in addition that (7.3) holds. Such t comprise a set of full measure in the range of u. Then $\partial\{u>t\}=\check{B}\cap\{u=t\}\cup\partial\check{B}\cap\partial\{u>t\}$ and $\{u>t\}$ is a Caccioppoli set in $\check{B}$ with finite $\check{g}$-perimeter; and likewise for $u^\star$. From Corollary 6.11,

$$\begin{aligned}\infty>\int_{\check{B}\cap\{u=t\}}\check{g}\,d\mathcal{H}^0&=\int_{\partial\{u>t\}}\check{g}\,d\mathcal{H}^0=P_{\check{g}}(\{u>t\},\check{B})\\&\geq P_{\check{g}}(\{u^\star>t\},\check{B})=\int_{\check{B}\cap\{u^\star=t\}}\check{g}\,d\mathcal{H}^0.\end{aligned} \tag{7.4}$$

Moreover, $\check{B}\cap\{u^\star=t\}$ consists of a singleton; thus,

$$\begin{aligned}\check{\mu}_{u^\star}'(t)&=-\int_{(\check{B}\setminus Z_\star)\cap\{u^\star=t\}}\frac{\check{g}}{|(u^\star)'|}\,d\mathcal{H}^0\\&=-\int_{\check{B}\cap\{u^\star=t\}}\frac{\check{g}}{|(u^\star)'|}\,d\mathcal{H}^0=-\int_{\check{B}\cap\{u^\star=t\}}\check{g}\,d\mathcal{H}^0\Big/\int_{\check{B}\cap\{u^\star=t\}}|(u^\star)'|\,d\mathcal{H}^0.\end{aligned} \tag{7.5}$$

By (7.3) and (7.5),

$$\begin{aligned}\int_{(\check{B}\setminus Z)\cap\{u=t\}}\frac{u^2+|u'|^2}{|u'|}\check{g}\,d\mathcal{H}^0&=\int_{(\check{B}\setminus Z)\cap\{u=t\}}\frac{u^2+|u'|^2}{|u'|}\check{g}\,d\mathcal{H}^0\\&=\int_{(\check{B}\setminus Z)\cap\{u=t\}}\left\{t^2+|u'|^2\right\}\frac{\check{g}\,d\mathcal{H}^0/|u'|}{-\check{\mu}_u'(t)}\Big(-\check{\mu}_u'(t)\Big)\\&\geq\left\{t^2+\left(\frac{\int_{(\check{B}\setminus Z)\cap\{u=t\}}\check{g}\,d\mathcal{H}^0}{-\check{\mu}_u'(t)}\right)^2\right\}\left(-\check{\mu}_u'(t)\right)\\&=\left\{t^2+\left(\frac{\int_{\check{B}\cap\{u=t\}}\check{g}\,d\mathcal{H}^0}{-\check{\mu}_u'(t)}\right)^2\right\}\left(-\check{\mu}_u'(t)\right)\end{aligned}$$

$$\geq \left\{ t^2 + \left(\frac{\int_{\check{B}\cap\{u^\star=t\}} \check{g}\, d\mathcal{H}^0}{-\check{\mu}'_{u^\star}(t)} \right)^2 \right\} \left(-\check{\mu}'_{u^\star}(t)\right)$$

$$= \int_{\check{B}\cap\{u^\star=t\}} \frac{(u^\star)^2 + |(u^\star)'|^2}{|(u^\star)'|} \check{g}\, d\mathcal{H}^0 \tag{7.6}$$

where Jensen's inequality has been used in the first inequality and (7.4) in the second. This inequality combined with (7.2) as well as Lemma 7.7 lead to the result. □

Corollary 7.9 *Let $u \in D(\check{\mathscr{E}})$. Then $u^\star \in D(\check{\mathscr{E}})$ and $\check{\mathscr{E}}(u, u) \geq \check{\mathscr{E}}(u^\star, u^\star)$.*

Proof Let $u \in D(\check{\mathscr{E}})$. By Lemma 5.5 we may choose a sequence (u_h) in $C^\infty(\overline{\check{B}})$ that converges to u in $(D(\check{\mathscr{E}}), \check{\mathscr{E}})$; each u_h is Lipschitz continuous on $\check{B}$. By Theorem 6.13, $(u_h^\star)$ converges to $u^\star$ in $L^2(\check{B}, \check{V})$. By Theorem 7.8, each $u_h^\star \in D(\check{\mathscr{E}})$ and $\check{\mathscr{E}}(u_h^\star, u_h^\star) \leq \check{\mathscr{E}}(u_h, u_h)$, so the sequence $(\check{\mathscr{E}}(u_h^\star, u_h^\star))_h$ is uniformly bounded in $\mathbb{R}$. By the Banach-Alaoglu theorem (cf. [19, A2 Theorem 2.1]) we may assume that $u_h^\star \to v$ weakly as $h \to \infty$ in $(D(\check{\mathscr{E}}), \check{\mathscr{E}})$ for some $v \in D(\check{\mathscr{E}})$ by selecting a subsequence if necessary. We may identify v with $u^\star$ thanks to the $L^2(\check{B}, \check{V})$ convergence and the Banach-Saks theorem (cf. [19, A2 Theorem 2.2]); hence $u^\star \in D(\check{\mathscr{E}})$. By [16, Theorem 10.1.5],

$$\check{\mathscr{E}}(u^\star, u^\star) \leq \liminf_{h\to\infty} \check{\mathscr{E}}(u_h^\star, u_h^\star) \leq \liminf_{h\to\infty} \check{\mathscr{E}}(u_h, u_h) = \check{\mathscr{E}}(u, u).$$

□

Corollary 7.10 *Let $u \in D(\hat{\mathscr{E}})$. Then $u^\bigstar \in D(\hat{\mathscr{E}})$ and $\hat{\mathscr{E}}(u, u) \geq \hat{\mathscr{E}}(u^\bigstar, u^\bigstar)$.*

Proof Let $u \in D(\hat{\mathscr{E}})$. Then $v := u \circ R^{-1} \in D(\check{\mathscr{E}})$ by Lemma 5.5. Morever, $u^\bigstar = v^\star \circ R$ by Proposition 6.10. By Lemma 5.5 and Corollary 7.9,

$$\hat{\mathscr{E}}(u, u) = \check{\mathscr{E}}(v, v) \geq \check{\mathscr{E}}(v^\star, v^\star) = \hat{\mathscr{E}}(v^\star \circ R, v^\star \circ R) = \hat{\mathscr{E}}(u^\bigstar, u^\bigstar).$$

□

8 Equality Case in the Pólya–Szegö Inequality

We now investigate the equality case in the Pólya–Szegö inequality.

Lemma 8.1 *Let $u \in D(\check{\mathscr{E}})$ and $t \in \mathbb{R}$.*

(i) *Put $v := u \wedge t$. Then $\int_{\check{B}} |v'|^2\, d\check{V} = \int_{\check{B}\cap\{u>t\}} |u'|^2\, d\check{V} = \int_{\check{B}\cap\{u\geq t\}} |u'|^2\, d\check{V}$.*
(ii) *Put $v := u \vee t$. Then $\int_{\check{B}} |v'|^2\, d\check{V} = \int_{\check{B}\cap\{u<t\}} |u'|^2\, d\check{V} = \int_{\check{B}\cap\{u\leq t\}} |u'|^2\, d\check{V}$.*

Proof We only prove (i). Write

$$\int_{\check{B}} |v'|^2\, d\check{V} = \int_{\check{B}\cap\{u<t\}} |v'|^2\, d\check{V} + \int_{\check{B}\cap\{u=t\}} |v'|^2\, d\check{V} + \int_{\check{B}\cap\{u>t\}} |v'|^2\, d\check{V}.$$

The set $\{u < t\}$ is open in $\check{B}$ as u is continuous so $v = u$ and $v' = u'$ there. By [13, Lemma 7.7], $v' = 0$ $\mathscr{L}^1$-a.e. on $\{u \geq t\}$. $\square$

Suppose that $u \in W^{1,2}_{\text{loc}}(\check{B})$ is precisely represented in the sense of [18, (2.5)]. Then the set $\{u = t\}$ is finite or countably infinite for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$ and the coarea formula (7.1) holds for u by [18, Theorem 1.1]. With Z as before it follows that $Z \cap \{u = t\} = \emptyset$ for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$ and hence $N := u(Z) \subset \mathbb{R}$ is $\mathscr{L}^1$-negligible.

Lemma 8.2 *Let u be a nonnegative function in $W^{1,2}_{\text{loc}}(\check{B})$ precisely represented in the sense of* [18, (2.5)]. *Then statements (i)–(vii) of Lemma* 7.6 *hold.*

Proof This runs as in Lemma 7.6. $\square$

Lemma 8.3 *Let $u \in D(\check{\mathscr{E}})$ be nonnegative. Then $\int_{\check{B}\cap Z} u^2\, d\check{V} = \int_{\check{B}\cap Z_\star} (u^\star)^2\, d\check{V}$.*

Proof The proof proceeds as in Lemma 7.7. $\square$

Lemma 8.4 *Let $u \in D(\check{\mathscr{E}})$ be nonnegative. Then*

(i) *for $t', t'' \in \mathbb{R}$ with $0 \leq t' < t''$,*

$$\int_{\check{B}\cap\{t'<u\leq t''\}} \left(u^2 + |u'|^2\right) d\check{V} \geq \int_{\check{B}\cap\{t'<u\leq t''\}} \left((u^\star)^2 + |(u^\star)'|^2\right) d\check{V};$$

(ii) *for $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$,*

$$\int_{(\check{B}\setminus Z)\cap\{\tilde{u}=t\}} \frac{\tilde{u}^2 + |\tilde{u}'|^2}{|\tilde{u}'|} \check{g}\, d\mathcal{H}^0 \geq \int_{(\check{B}\setminus Z_\star)\cap\{u^\star=t\}} \frac{(u^\star)^2 + |(u^\star)'|^2}{|(u^\star)'|} \check{g}\, d\mathcal{H}^0$$

where $\tilde{u}$ is the unique continuous representative of u (cf. [1, Definition 3.31 and after]*).*

Proof (i) Put $v := (u \vee t') \wedge t'' \in D(\check{\mathscr{E}})$. By Lemma 8.1,

$$\int_{\check{B}} \left(v^2 + |v'|^2\right) d\check{V} = \int_{\check{B}\cap\{t'<u\leq t''\}} \left(u^2 + |u'|^2\right) d\check{V} + t'^2(1 - \mu_u(t')) + t''^2 \mu_u(t'').$$

By Lemma 10.2, $\check{v}^\sharp = (\check{u}^\sharp \vee t') \wedge t''$ on $[0, \check{V}(\check{B})]$ and hence $v^\star = (u^\star \vee t') \wedge t''$. We may then write an identity of the above form but with $u^\star$, $v^\star$ in place of u, v. The statement then follows from Corollary 7.9.

(ii) Note that $\tilde{u}$ is precisely represented in the sense of [18, (2.5)]. By the coarea formula for Sobolev mappings [18, Theorem 1.1],

$$\int_{\check{B}} \phi |u'|\, dx = \int_{-\infty}^{\infty} \int_{\check{B}\cap\{\tilde{u}=t\}} \phi\, d\mathcal{H}^0\, dt$$

for any $\mathscr{L}^1$-measurable function $\phi : \check{B} \to [0, \infty]$. In particular,

$$\int_{\check{B}\setminus Z} \phi(u)\left(u^2 + |u'|^2\right) d\check{V} = \int_{-\infty}^{\infty} \int_{(\check{B}\setminus Z)\cap\{\tilde{u}=t\}} \left\{ \frac{\tilde{u}^2 + |\tilde{u}'|^2}{|\tilde{u}'|} \check{g}\, d\mathcal{H}^0 \right\} \phi(t)\, dt \quad (8.1)$$

for any $\mathscr{L}^1$-measurable function $\phi : \mathbb{R} \to [0, \infty]$.
Define

$$w(t) := \int_{(\check{B} \setminus Z) \cap \{u > t\}} (u^2 + |u'|^2) \, d\check{V}$$

for $t \in \mathbb{R}$. For $\varphi \in C_c^\infty(\mathbb{R}, \mathbb{R})$,

$$\int_{-\infty}^{\infty} w\varphi' \, dx = \int_{\check{B} \setminus Z} \varphi(u)(u^2 + |u'|^2) \, d\check{V}$$

so that $w \in \mathrm{BV}(\mathbb{R})$. By (8.1), $Dw = \rho \mathscr{L}^1$ where

$$\rho(t) := - \int_{(\check{B} \setminus Z) \cap \{\tilde{u} = t\}} \frac{\tilde{u}^2 + |\tilde{u}'|^2}{|\tilde{u}'|} \check{g} \, d\mathcal{H}^0$$

for $t \in \mathbb{R}$; that is, w is absolutely continuous [1, Definition 3.31]. By [1, Theorem 3.28], w is differentiable $\mathscr{L}^1$-a.e. on $\mathbb{R}$ and $w' = \rho$ $\mathscr{L}^1$-a.e. on $\mathbb{R}$. The same holds for the function $w_\star$ defined as for w but with $u^\star$ in place of u. Note that $\widetilde{u^\star} = u^\star$. The statement then follows from (i). □

We state the following lemma without proof.

Lemma 8.5 *Let u be a continuous real-valued function on $\check{B}$. Suppose that for $\mathscr{L}^1$-a.e. $t > 0$ the set $\{u > t\}$ is either an open interval in $\check{B}$ abutting a boundary point or $\{u > t\} = \emptyset$. Then u is monotone on $\check{B}$.*

Theorem 8.6 *Let $u \in D(\check{\mathscr{E}})$ be nonnegative and suppose that $\check{\mathscr{E}}(u, u) = \check{\mathscr{E}}(u^\star, u^\star)$. Then $\tilde{u}$ is monotone on $\check{B}$.*

Proof By (8.1),

$$\check{\mathscr{E}}(u, u) = \int_{\check{B} \cap Z} u^2 \, d\check{V} + \int_{\mathrm{Ran}(\tilde{u})} \int_{(\check{B} \setminus Z) \cap \{\tilde{u} = t\}} \frac{\tilde{u}^2 + |\tilde{u}'|^2}{|\tilde{u}'|} \check{g} \, d\mathcal{H}^0 \, dt$$

and a similar identity holds for $u^\star$. We may assume that $\mathrm{Ran}(\tilde{u})$ is a closed interval in $[0, \infty)$ with non-empty interior. By Lemmas 8.3 and 8.4,

$$\int_{(\check{B} \setminus Z) \cap \{\tilde{u} = t\}} \frac{\tilde{u}^2 + |\tilde{u}'|^2}{|\tilde{u}'|} \check{g} \, d\mathcal{H}^0 = \int_{(\check{B} \setminus Z_\star) \cap \{u^\star = t\}} \frac{(u^\star)^2 + |(u^\star)'|^2}{|(u^\star)'|} \check{g} \, d\mathcal{H}^0$$

for $\mathscr{L}^1$-a.e. $t \in \mathrm{Ran}(\tilde{u})$. The chain of inequalities in (7.6) is valid with $\tilde{u}$ in place of u and we may replace the sign $\geq$ with the equality sign. In particular,

$$\int_{\check{B} \cap \{\tilde{u} = t\}} \check{g} \, d\mathcal{H}^0 = \int_{\check{B} \cap \{u^\star = t\}} \check{g} \, d\mathcal{H}^0 \tag{8.2}$$

for $\mathscr{L}^1$-a.e. $t \in \operatorname{Ran}(\tilde{u})$. Suppose that $t \in \operatorname{Ran}(\tilde{u})$ such that $Z \cap \{\tilde{u} = t\} = \emptyset$ and $\int_{\check{B}\cap\{\tilde{u}=t\}} \check{g}\, d\mathcal{H}^0 < \infty$. We remark that $\partial\{\tilde{u} > t\} = \check{B} \cap \{\tilde{u} = t\} \cup \partial\check{B} \cap \partial\{\tilde{u} > t\}$ and $\{\tilde{u} > t\}$ is a Caccioppoli set in $\check{B}$ with finite $\check{g}$-perimeter $P_{\check{g}}(\{\tilde{u} > t\}, \check{B}) = \int_{\check{B}\cap\{\tilde{u}=t\}} \check{g}\, d\mathcal{H}^0 < \infty$. From the identity (8.2) we derive that $P_{\check{g}}(\{\tilde{u} > t\}, \check{B}) = P_{\check{g}}(\{u^\star > t\}, \check{B})$ for $\mathscr{L}^1$-a.e. $t \in \operatorname{Ran}(\tilde{u})$. By Corollary 6.12 the set $\{\tilde{u} > t\}$ is either an open interval in $\check{B}$ abutting a boundary point or $\{\tilde{u} > t\} = \emptyset$ for $\mathscr{L}^1$-a.e. $t > 0$. The statement follows by Lemma 8.5. □

Corollary 8.7 *Let $u \in D(\hat{\mathscr{E}})$ be nonnegative and suppose that $\hat{\mathscr{E}}(u, u) = \hat{\mathscr{E}}(u^\bigstar, u^\bigstar)$. Then $\tilde{u}$ is monotone on B.*

Proof Put $v := u \circ R^{-1} \in D(\check{\mathscr{E}})$ by Lemma 5.5. Also, $u^\bigstar = v^\star \circ R$ by Proposition 6.10. We have

$$\hat{\mathscr{E}}(u, u) = \check{\mathscr{E}}(v, v) \geq \check{\mathscr{E}}(v^\star, v^\star) = \hat{\mathscr{E}}(v^\star \circ R, v^\star \circ R) = \hat{\mathscr{E}}(u^\bigstar, u^\bigstar).$$

by Lemma 5.5 and Corollary 7.9. So $\check{\mathscr{E}}(v, v) = \check{\mathscr{E}}(v^\star, v^\star)$. By Theorem 8.6, $\tilde{v}$ is monotone on $\check{B}$ and hence $\tilde{u}$ is monotone on B. □

9 Application to Exchange Flow

Consider the positive definite bilinear form $(\mathscr{D}, \mathscr{E})$ in $L^2(B, \mathscr{L}^1)$ given by

$$\mathscr{E}(u, v) := \int_B u'v'\, dx \qquad \left(u, v \in \mathscr{D} := \left\{u : u/\psi \in C^\infty(\overline{B})\right\}\right).$$

Here, $\psi := u_B$ is given by $\psi(x) = (1/2)(1 - |x|^2)$ for $x \in B$.

Lemma 9.1 (i) *$(\mathscr{D}, \mathscr{E})$ is closable in $L^2(B, \mathscr{L}^1)$ with closure denoted $(D(\mathscr{E}), \mathscr{E})$;*
(ii) *$(D(\mathscr{E}), \mathscr{E})$ is a symmetric Dirichlet form in $L^2(B, \mathscr{L}^1)$;*
(iii) *$D(\mathscr{E}) = W_0^{1,2}(B)$.*

Proof (i) Suppose (u_h) is a sequence in $\mathscr{D}$ such that $u_h \to 0$ in $L^2(B, \mathscr{L}^1)$. We write $v \in \mathscr{D}$ in the form $v = \psi w$ for some $w \in C^\infty(\overline{B})$; so $v'' \in L^2(B, \mathscr{L}^1)$. An integration-by-parts gives

$$\int_B u_h' v'\, dx = \int_{\partial B} u_h v' \nu\, d\mathcal{H}^0 - \int_B u_h v''\, dx = -\int_B u_h v''\, dx \to 0$$

as $h \to \infty$ where $\nu = \pm 1$ is the unit exterior normal on ∂B. The statement follows by [19, Lemma I.3.4]. Then $(D(\mathscr{E}), \mathscr{E})$ is a symmetric closed form by definition (cf. [19, Definition I.2.3]). By [19, Proposition I.4.10 and II.2 (c)], $(D(\mathscr{E}), \mathscr{E})$ is a symmetric Dirichlet form and (ii) follows. Note that $\psi = u_B \in W_0^{1,2}(B)$. Thus $C_0^\infty(B) \subset \mathscr{D} \subset W_0^{1,2}(B)$. This proves (iii). □

The transient Dirichlet space $(D(\mathscr{E}), \mathscr{E})$ has reference function ψ^{-1} and

$$\int_B |u|\psi^{-1}\, dx \le \sqrt{\mathscr{E}(u,u)} \text{ for all } u \in D(\mathscr{E}). \tag{9.1}$$

Denote by $D(\mathscr{E})_e$ the extended Dirichlet space; that is, the family of $\mathscr{L}^1$-measurable functions u on B such that $|u| < \infty$ $\mathscr{L}^1$-a.e. and there exists an $\mathscr{E}$-Cauchy sequence (u_h) of functions in $D(\mathscr{E})$ such that $u_h \to u$ $\mathscr{L}^1$-a.e. on B. By [12, Lemma 1.5.5], $(D(\mathscr{E})_e, \mathscr{E})$ is a Hilbert space. The identity (9.1) extends to $D(\mathscr{E})_e$ and $D(\mathscr{E})_e \subset L^1(B, \psi^{-1}\mathscr{L}^1)$.
In the notation of Sect. 5 we take $f = \psi$ and $g = \psi^{3/2}$. The conditions (A.1)–(A.5) are satisfied by Proposition 6.5. Let V be the measure $V := \psi\mathscr{L}^1$ on B. We work with the symmetric Dirichlet form

$$\hat{\mathscr{E}}(u,v) := \int_B \Big(uv + \psi u'v'\Big)\, dV \qquad (u, v \in D(\hat{\mathscr{E}}))$$

in $L^2(B, V)$ with domain

$$D(\hat{\mathscr{E}}) := \Big\{u \in L^2(B,V) : u \text{ is weakly differentiable on } B \text{ and } \psi^{1/2}u' \in L^2(B,V)\Big\}.$$

Proposition 9.2 *The bijective mapping $\mathscr{D} \to C^\infty(\overline{B})$; $u \mapsto \overline{u} := u/\psi$ extends to a Hilbert space isomorphism $E : D(\mathscr{E})_e \to D(\hat{\mathscr{E}})$. In particular, for $u, v \in D(\mathscr{E})_e$ we have that*

$$\mathscr{E}(u,\, v) = \hat{\mathscr{E}}(\overline{u},\, \overline{v}). \tag{9.2}$$

Proof Using integration-by-parts and the fact that $-\psi'' = 1$ on B gives

$$\begin{aligned}
\int_B \overline{u}'\, \overline{v}'\, \psi^2\, dx &= \int_B (u/\psi)'\, \overline{v}'\, \psi^2\, dx\\
&= \int_B \Big(\psi\, u' - u\, \psi'\Big)\, \overline{v}'\, dx\\
&= \int_B u'\ \overline{v}'\ \psi\, dx - \int_{\partial B} u\, \psi'\, \overline{v}\, \nu\, d\mathcal{H}^0 + \int_B (u\, \psi')'\overline{v}\, dx\\
&= \int_B u'\, \overline{v}'\ \psi\, dx + \int_B (u\, \psi')'\overline{v}\, dx\\
&= \int_B u'\, \overline{v}'\ \psi\, dx + \int_B (u'\ \psi')\, \overline{v} - u\, \overline{v}\, dx\\
&= \int_B u'\, v'\, dx - \int_B \overline{u}\, \overline{v}\, \psi\, dx
\end{aligned}$$

where $\nu = \pm 1$ is the unit exterior normal on ∂B. This establishes (9.2) on $\mathscr{D}$. The map E extends to $D(\mathscr{E})$ by density of $\mathscr{D}$ in $D(\mathscr{E})$ as does (9.2). Let $u \in D(\mathscr{E})_e$ and choose a sequence (u_h) in $D(\mathscr{E})$ such that $u_h \to u$ $\mathscr{L}^1$-a.e. on B as $h \to \infty$ and (u_h) is a $\mathscr{E}$-Cauchy sequence. Then $(\overline{u}_h)$ is a Cauchy sequence in $(D(\hat{\mathscr{E}}), \hat{\mathscr{E}})$ with limit

$v \in D(\hat{\mathscr{E}})$ say. Define $Eu := v$. This definition is well-defined and the identity (9.2) holds on $D(\mathscr{E})_e$. In particular, the mapping E is injective.
We show that E is a surjection. Let $v \in D(\hat{\mathscr{E}})$. We put $u := \psi v$ and claim that $u \in D(\mathscr{E})_e$. Then there exists a sequence (w_h) in $C^\infty(\overline{\check{B}})$ such that $v_h := w_h \circ R \to v$ in $(D(\hat{\mathscr{E}}), \hat{\mathscr{E}})$ as $h \to \infty$. Put $u_h := \psi v_h$ for each h. Now $u_h \in C^1(\overline{B})$ and $u_h = 0$ on ∂B for each h. This means that (u_h) is a sequence in $D(\mathscr{E})$. By (9.2), (u_h) is a $\mathscr{E}$-Cauchy sequence. As (v_h) converges to v in $L^2(B, V)$ we may assume that (v_h) converges to v V-a.e. on B by selecting a subsequence if necessary. Thus, (u_h) converges to u $\mathscr{L}^1$-a.e. on B. This shows that $u \in D(\mathscr{E})_e$ and $Eu = v$. □

Theorem 9.3 *For any $\mathscr{L}^1$-measurable set A in B, $J(A) \leq J(A^\bigstar)$.*

Proof Let A be an $\mathscr{L}^1$-measurable set in B and put $u := u_A \in D(\mathscr{E})$ and $v := u_{A^\star} \in D(\mathscr{E})$. By Theorem 6.6, Proposition 9.2, the Cauchy–Schwarz inequality and Corollary 7.10,

$$\begin{aligned} J(A) = \int_B \overline{u}\, \chi_A \, dV &\leq \int_B \overline{u}^\bigstar \chi_{A^\star} \, dV = \mathscr{E}(\psi \overline{u}^\bigstar, v) = \hat{\mathscr{E}}(\overline{u}^\bigstar, \overline{v}) \\ &\leq \hat{\mathscr{E}}(\overline{u}^\bigstar, \overline{u}^\bigstar)^{1/2} \hat{\mathscr{E}}(\overline{v}, \overline{v})^{1/2} \leq \hat{\mathscr{E}}(\overline{u}, \overline{u})^{1/2} \hat{\mathscr{E}}(\overline{v}, \overline{v})^{1/2} = J(A)^{1/2} J(A^\bigstar)^{1/2} \end{aligned}$$

and the result follows. □

Theorem 9.4 *Suppose that A is an $\mathscr{L}^1$-measurable set in B such that $J(A) = J(A^\bigstar)$. Then A is $\mathscr{L}^1$-a.e. equivalent to an open interval in B abutting a boundary point of B.*

Proof We may assume that $0 < V(A) < V(B)$. From the chain of inequalities in the proof of Theorem 9.3 we derive that $\hat{\mathscr{E}}(\overline{u}^\bigstar, \overline{u}^\bigstar) = \hat{\mathscr{E}}(\overline{u}, \overline{u})$. Put $v := \overline{u}$. From Corollary 8.7 we infer that $\widetilde{v}$ is monotone on B. Put $p := V(A)/V(B) \in (0, 1)$. By Theorem 9.3, A is an optimal configuration for the problem (1.2) for the data (B, p). By Proposition 3.3, $V(A \Delta \{\widetilde{v} > c\}) = 0$ for some $c \in (0, 1)$. As $\widetilde{v}$ is monotone, $\{\widetilde{v} > c\}$ is an open interval in B abutting an end-point of B. This leads to the result. □

We may now characterise optimal configurations for the problem (1.2).

Theorem 9.5 *Let $p \in (0, 1)$.*

(i) *The sets $(-1, F^{-1}(pV(B)))$ and $(F^{-1}((1-p)V(B)), 1)$ are optimal configurations for the problem (1.2) with data (B, p).*
(ii) *If $E \subset B$ is an optimal configuration for the data (B, p) then E is $\mathscr{L}^1$-a.e. equivalent to one of the sets in (i).*

Let U be a bounded open connected set in $\mathbb{R}^n$ $(n \geq 1)$.

Proposition 9.6 (i) *For $\lambda \in (-1, 1)$, $\gamma(U, \lambda) = 2I(U, p)$ where $p = (1-\lambda)/2$.*
(ii) $\gamma(U) = 2 \sup_{p \in (0,1)} I(U, p)$.

Proof (i) Let A be an open set in U. Suppose u satisfies (1.3) and the condition $(u, 1) = 0$. Put $f := -(\lambda + 1)\chi_A - (\lambda - 1)\chi_{U\setminus A}$; then $u = Gf$. From the flux-balance condition and symmetry of the Green operator,

$$0 = (u, 1) = (\psi, f) = -(\lambda + 1)V(A) - (\lambda - 1)V(U \setminus A).$$

So $\lambda = V(U)^{-1}\{V(U \setminus A) - V(U)\}$ and $V(A) = \frac{1-\lambda}{2}V(U)$. We have

$$\begin{aligned}
(\chi_{U\setminus A}, u) &= (G\chi_{U\setminus A}, f)\\
&= -(\lambda + 1)(G\chi_{U\setminus A}, \chi_A) - (\lambda - 1)(G\chi_{U\setminus A}, \chi_{U\setminus A})\\
&= -(\lambda + 1)(V(A) - J(A)) - (\lambda - 1)(V(U \setminus A) - V(A) + J(A))\\
&= 2J(A) - (\lambda + 1)V(A) - (\lambda - 1)V(U \setminus A) + (\lambda - 1)V(A)\\
&= 2J(A) + (\lambda - 1)V(A) = 2\Big\{J(A) - V(U)p^2\Big\}
\end{aligned}$$

where $p = (1 - \lambda)/2$. This leads to the reformulation (i). (ii) follows immediately. □

Recall that for $x, y \in B$,

$$G(x, y) = \begin{cases} (1/2)(1 - y)(1 + x) & \text{for } x \le y;\\ (1/2)(1 + y)(1 - x) & \text{for } x \ge y. \end{cases}$$

Lemma 9.7 *The mapping* $(0, 1) \to \mathbb{R}$; $p \to I(B, p)$ *has a unique global maximum at* $p = 1/2$.

Proof Define $\eta : B \to \mathbb{R}$ by $\eta = J(A) - V(B)^{-1}V(A)^2$ where $A = (-1, t)$. Then $I(B, p) = \eta(F^{-1}(pV(B)))$ for $p \in (0, 1)$ by Theorem 9.5. For $t \in B$,

$$\begin{aligned}
\frac{d}{dt}J((-1, t)) &= \frac{d}{dt}\int_{-1}^{t} G\chi_{(-1,t)}(x)\,dx = G\chi_{(-1,t)}(t) + \int_{-1}^{t}\frac{d}{dt}G\chi_{(-1,t)}(x)\,dx\\
&= G\chi_{(-1,t)}(t) + \int_{-1}^{t} G(x, t)\,dx = 2G\chi_{(-1,t)}(t),
\end{aligned}$$

so that

$$\eta'(t) = 2G\chi_{(-1,t)}(t) - 2V(B)^{-1}V((-1, t))\psi(t).$$

A further computation gives

$$G\chi_{(-1,t)}(\xi) = (1/4)(1 - t)(1 + t)^2 \text{ and } V(B)^{-1}V((-1, t)) = (1/4)(2 + 3t - t^3).$$

We then obtain that $\eta'(t) = -(1/4)t(1 - t^2)^2$ for any $t \in B$. This proves the result.□

Theorem 9.8 *Let* $\lambda \in (-1, 1)$.

(i) *The sets* $(-1, F^{-1}(pV(B)))$ *and* $(F^{-1}((1 - p)V(B)), 1)$ *are optimal configurations for the problem* (1.5) *with data* (U, λ) *where* $p = (1 - \lambda)/2$.

(ii) *If $E \subset B$ is an optimal configuration for the problem* (1.5) *with data* (U, λ) *then E is $\mathscr{L}^1$-a.e. equivalent to one of the sets in (i).*
(iii) *The sets* $(-1, 0)$ *and* $(0, 1)$ *are optimal configurations for the problem* (1.4).
(iv) *If $E \subset B$ is an optimal configuration for the problem* (1.4) *then E is $\mathscr{L}^1$-a.e. equivalent to one of the sets in (iii).*

Proof Parts (i) and (ii) follow from Proposition 9.6 and Theorem 9.5. (iii) and (iv) follow from Proposition 9.6 and Lemma 9.7. □

Appendix: On the Generalised Inverse Function

Let $(X, \mathscr{A}, \mu)$ be a finite measure space. Let $f : X \to [0, \infty)$ be an $\mathscr{A}$-measurable function. The distribution function $\mu_f : [0, \infty) \to [0, \mu(X)]$ of f is defined by $\mu_f(t) := \mu(\{f > t\})$ for $t \geq 0$. Note that μ_f is right-continuous and non-increasing on $[0, \infty)$ and $\mu_f(t) \to 0$ as $t \to \infty$. The generalised inverse $f^\sharp : [0, \mu(X)] \to [0, \infty]$ of μ_f is defined by $f^\sharp(s) := \inf\{t \geq 0 : \mu_f(t) \leq s\}$ with the understanding that $\inf \emptyset = +\infty$.

Lemma 10.1 *Let $f : X \to [0, \infty)$ be an $\mathscr{A}$-measurable function. Let $t \geq 0$ and $s \in [0, \mu(X)]$. Then*

(i) *$\mu_f(t) > s$ if and only if $f^\sharp(s) > t$;*
(ii) *$\mu_f(t) \leq s$ if and only if $f^\sharp(s) \leq t$;*
(iii) *$\mu_f(t-) < s$ then $f^\sharp(s) < t$ in case $t > 0$.*

Proof Suppose $\mu_f(t) > s$. The set $\{\mu_f > s\} \subset [0, \infty)$ is relatively open by right-continuity of μ_f. It follows that $f^\sharp(s) > t$. Conversely, if $f^\sharp(s) > t$ then $\mu_f(t) > s$ be definition of $f^\sharp$. This shows *(i)*. (ii) follows from (i). Suppose $t > 0$ and $\mu_f(t-) < s$. By definition of the left-handed limit, we may choose $0 < \tau < t$ such that $\mu_f(t-) \leq \mu_f(\tau) < s$. Then $f^\sharp(s) \leq \tau < t$. This shows (iii). □

Lemma 10.2 *Let $f : X \to [0, \infty)$ be an $\mathscr{A}$-measurable function and $t > 0$. Then*

(i) *if $g := f \wedge t$ then $g^\sharp = f^\sharp \wedge t$ on $[0, \mu(X))$;*
(ii) *if $g := f \vee t$ then $g^\sharp = f^\sharp \vee t$ on $[0, \mu(X))$.*

Proof (i) We have that $\mu_g = \chi_{[0,t)}\mu_f$ on $[0, \infty)$ and $g^\sharp = t\chi_{[0,\mu_f(t-))} + f^\sharp\chi_{[\mu_f(t-),\mu(X))}$. On the other hand, $w := f^\sharp \wedge t = f^\sharp\chi_{\{f^\sharp<t\}} + t\chi_{\{f^\sharp \geq t\}}$. Let $0 \leq s < \mu_f(t-)$. As $\mu_f \geq \mu_f(t-) > s$ on $[0, t)$, $\{\mu_f \leq s\} \subset [t, \infty)$ and $f^\sharp(s) \geq t$; that is, $w(s) = t = g^\sharp(s)$. Now suppose that $s = \mu_f(t-)$. Note that $f^\sharp(s) \leq t$. Consider the case that $f^\sharp(s) = t$. Then $w(s) = t = f^\sharp(s) = g^\sharp(s)$. If $f^\sharp(s) < t$

then $w(s) = f^\sharp(s) = g^\sharp(s)$. Finally, suppose that $s > \mu_f(t-)$. By Lemma 10.1 (iii), $w(s) = f^\sharp(s) = g^\sharp(s)$. Item (i) follows.

(ii) In this case, $\mu_g = \chi_{[0,t)} + \chi_{[t,\infty)}\mu_f$ on $[0,\infty)$ and $g^\sharp = \chi_{[0,\mu_f(t))}f^\sharp + t\chi_{[\mu_f(t),\mu(X))}$. On the other hand, $w := f^\sharp \vee t = t\chi_{\{f^\sharp \le t\}} + f^\sharp\chi_{\{f^\sharp > t\}}$. Let $0 \le s < \mu_f(t)$. By Lemma 10.1 (i), $f^\sharp(s) > t$ and $w(s) = f^\sharp(s) = g^\sharp(s)$. Now suppose that $s \ge \mu_f(t)$. By Lemma 10.1 (ii), $f^\sharp(s) \le t$ and $w(s) = t = g^\sharp(s)$. □

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